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Repeated spatial extrapolation: An extraordinarily efficient approach for option pricing

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HIGHLIGHTS

- The repeated spatial extrapolation yields extraordinarily efficient approximations of European vanilla and digital option prices.
- The repeated spatial extrapolation has not been used so far for option pricing.
- We show that the repeated spatial extrapolation achieves superior accuracy even if the final solutions are discontinuous.
- The repeated spatial extrapolation neatly outperforms the already existing finite difference approaches.

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ABSTRACT

Various finite difference methods for option pricing have been proposed. In this paper we demonstrate how a very simple approach, namely the repeated spatial extrapolation, can perform extremely better than the finite difference schemes that have been developed so far. In particular, we consider the problem of pricing vanilla and digital options under the Black–Scholes model, and show that, if the payoff functions are dealt with properly, then errors close to the machine precision are obtained in only some hundredths of a second.

1. Introduction

Several models for option pricing lead to partial differential equations in two independent variables, namely the time and the price of the underlying asset. Except for a few cases, such as, for example, the Black–Scholes equation for European Call and Put options [1], these partial differential equations do not have exact closed-form solutions, and must be solved by numerical approximation. This task is often accomplished by means of finite difference methods, which are very flexible and easy to implement.

In particular, the discretization along the price variable, hereafter referred to as *spatial* discretization, is often carried out by the standard (centered) three-point finite difference scheme, which, as shown by [2], is second-order accurate. Practically speaking, this means that, when the spatial discretization interval is halved, the error is reduced by a factor approximately equal to four.

In order to obtain more accurate solutions, some authors have proposed finite difference methods that are *formally* fourth-order accurate in space (see, for example, [3–8]). However, due to the fact that the options' payoffs, i.e., the final solutions of the partial differential problems being solved, are non-differentiable, these schemes achieve fourth-order accuracy only if they are used in conjunction with ad-hoc local mesh refinement techniques. Therefore, it is not clear if the above fourth-order methods can be proficiently applied to options with discontinuous payoffs (in such a case we should

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use a very irregular mesh that could spoil the accuracy of the overall numerical approximation), or if they can be further developed to obtain spatial convergence rates higher than four.

Spatial convergence rates higher than second-order are also experienced in [9–11], where the accuracy of the (centered) three-point finite difference scheme is enhanced by *spatial* extrapolation. Precisely, in these works a single step of the well-known Richardson extrapolation procedure is performed along the spatial direction, that is an improved solution is computed by extrapolation of solutions obtained using two different spatial discretization intervals (which allows one to remove the second-order component of the discretization error). Note that in [9,10] it is claimed that in order for the spatial extrapolation to be effective the options' payoffs must be replaced with infinitely smooth approximating functions. This is in contrast with results obtained in [2], where it is shown that the (centered) three-point finite difference scheme can achieve second-order accuracy even in the presence of non-smooth payoffs. Now, to the best of our knowledge, in the option pricing literature, the papers [9–11] are the only works in which the Richardson extrapolation procedure is applied along the price variable in conjunction with finite difference schemes (in the option pricing context the Richardson extrapolation has also been used along the time variable in conjunction with time discretization methods or with binomial and trinomial trees, see for example [12–15]).

Therefore, the following questions arise: (1) is it possible to further improve the accuracy of the three-point finite difference approximation by using *repeated* spatial extrapolation, i.e. by applying the spatial extrapolation for more than one time? The answer to this question is not straightforward, given that the partial differential problems to be solved have irregular final data, and, as already observed, finite difference schemes whose formal orders of convergence are higher that two turn out to be, in reality, second-order accurate only; (2) do the options' payoffs require a special smoothing procedure for the repeated spatial extrapolation to be successful?

The present paper wants to address the above two questions. In particular, we consider two popular benchmark problems, namely the pricing of a European vanilla Call option and the pricing of a European digital Call option under the Black–Scholes model, and solve them using the (centered) three-point finite difference scheme with repeated spatial extrapolation. The results obtained are excellent: the approximation error is dramatically reduced as the Richardson extrapolation procedure is repeated more and more times, and levels of accuracy close to machine precision are rapidly achieved. Moreover, to obtain such results the options' payoffs do not need to be replaced with infinitely smooth functions. On the contrary, following [2], it is sufficient to align the finite difference mesh with the payoff of the option being priced, and, in the case of the digital option only, to project the payoff to the space of continuous piecewise linear functions.

Another very interesting finding of this paper is that the repeated spatial extrapolation is extraordinarily efficient from the computational standpoint. In fact, as shown in Section 4, the option prices can be computed with errors of order 10^{-11} or 10^{-12} in only some hundredths of a second. To the best of our knowledge, such levels of performance are much higher than those that can be achieved using the lattice-based option pricing methods that have been developed so far.

Finally, let us observe that in this paper we have considered option pricing problems that have exact closed-form solutions. This choice is made in order to perform a direct and accurate estimation of the approximation error (without an exact closed-form solution it would be hard to estimate the errors obtained as they are sometimes of order 10^{-9} or even smaller, see Section 4). Nevertheless, the repeated spatial extrapolation could also be applied to a variety of models for which analytical solutions are not available (see Section 5 for a discussion on the possible extensions and limitations of the method).

The remainder of the paper is organized as follows: in Section 2 we show the partial differential problems that allow one to price European vanilla and digital Call options under the Black–Scholes model; in Section 3 we perform the numerical approximation of the problems described in Section 2; in particular the repeated spatial extrapolation procedure is developed in Section 3.3; in Section 4 we present the numerical results obtained using the proposed extrapolation approach; finally in Section 5 we summarize the main findings of this paper and examine possible future developments of the proposed approach.

2. Pricing European vanilla and digital Call options under the Black–Scholes model

Let $S(t)$ denote the price of an option's underlying asset as a function of time. According to the Black–Scholes model [1] $S(t)$ satisfies the following stochastic differential equations:

$$dS(t) = \mu S(t) + \sigma S(t) dW(t), \tag{1}$$

where μ and σ are positive constant parameters, referred to as drift and volatility, respectively, and W is a Wiener standard process (see [16]).

Let $V(S, t)$ denote the price of a European option on the above asset maturing at time T. The function $V(S, t)$ must satisfy the partial differential equation (the so-called Black–Scholes equation):

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0, \quad S \in (0, +\infty),\ t \in [0, T), \tag{2}$$

where r is the (constant) interest rate. Eq. (2) must be equipped with the final condition:

$$V(S, T) = \phi(S), \quad S \in [0, +\infty), \tag{3}$$