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Oscillatory behavior of third-order nonlinear delay dynamic equations on time scales

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ABSTRACT

In this paper, a class of third-order nonlinear delay dynamic equations on time scales is studied. By using the generalized Riccati transformation and the integral averaging technique, three new sufficient conditions which ensure that every solution is oscillatory or converges to zero are established. The results obtained essentially generalize and improve earlier ones.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger [1], in order to unify continuous and discrete analysis. Several authors have expounded on various aspects of this new theory; see [2–6]. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to plenty of applications, among them the study of population dynamic models which are discrete in season (and may follow a difference scheme with variable step-size or be modeled by continuous dynamic systems); they die out, say in winter, while their eggs are incubating or dormant, and then, in season again, hatching gives rise to a nonoverlapping population; see [4]. Not only does the new theory of so-called "dynamic equations" unify the theories of differential equations and difference equations, but it also extends these classical cases to cases "in between", e.g., to so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ for some $q > 1$ (which has important applications in quantum theory) and can be applied on different types of time scales such as $\mathbb{T} = h\mathbb{N}$, $\mathbb{T} = \mathbb{N}^2$ and the space of the harmonic numbers.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales, and we refer the reader to the studies by Bohner et al. [7] and Erbe et al. [8,9]. And there are some results dealing with oscillatory behavior of second-order delay dynamic equations on time scales [10–15]. However, there are few results dealing with the oscillation of the solutions of third-order delay dynamic equations on time scales; we refer the reader to the papers [16–18].

In this paper, we consider oscillatory behavior of all solutions of the third-order nonlinear delay dynamic equation

$$\left(r_2(t)\left[(r_1(t)x^{\Delta}(t))^{\Delta}\right]^{\alpha}\right)^{\Delta} + q(t)f(x[\tau(t)]) = 0, \quad t \in \mathbb{T},\ t \geq t_0, \tag{1.1}$$

where $\alpha \geq 1$ is the ratio of two positive odd integers.

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Throughout this paper, we will assume the following hypotheses.

(H_1) $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above, and $t_0 \in \mathbb{T}$ with $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

(H_2) $r_1(t), r_2(t), q(t)$ are positive, real-valued rd-continuous functions (i.e., functions are said to be rd-continuous if they are continuous at each right-dense point and if there exists a finite left limit at all left-dense points) defined on $\mathbb{T}$, and $r_1(t), r_2(t)$ satisfy

$$\int_{t_0}^{\infty} \frac{1}{r_1(s)} \triangle s = \infty, \qquad \int_{t_0}^{\infty} \left(\frac{1}{r_2(s)}\right)^{\frac{1}{\alpha}} \triangle s = \infty.$$

(H_3) $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing and differentiable function such that

$$\tau(t) \le t, \qquad \lim_{t\to\infty} \tau(t) = \infty, \quad \text{and} \quad \tau(\mathbb{T}) = \mathbb{T}.$$

(H_4) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\frac{f(x)}{x^\alpha} \ge K > 0$ for $x \ne 0$.

By a solution of (1.1), we mean a nontrivial function $x(t)$ satisfying (1.1) which has the properties $x(t) \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$ for $T_x \ge t_0$, and $r_2(t)\left[(r_1(t)x^\triangle(t))^\triangle\right]^\alpha \in C^1_{rd}([T_x, \infty)_{\mathbb{T}}, \mathbb{R})$. Our attention is restricted to those solutions of (1.1) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. A solution x of Eq. (1.1) is said to be oscillatory on $[T_x, \infty)_{\mathbb{T}}$ if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

If $\alpha = 1$, $\tau(t) = t$, then (1.1) simplifies to the third-order nonlinear dynamic equation

$$\left(r_2(t)\left[(r_1(t)x^\triangle(t))^\triangle\right]\right)^\triangle + q(t)f(x(t)) = 0, \quad t \in \mathbb{T},\ t \ge t_0. \tag{1.2}$$

If, furthermore, $r_1(t) = r_2(t) = 1$, $f(x) = x$, $\tau(t) = t$, then (1.1) reduces to the third-order linear dynamic equation

$$x^{\triangle\triangle\triangle}(t) + q(t)x(t) = 0, \quad t \in \mathbb{T},\ t \ge t_0. \tag{1.3}$$

If, in addition, $\alpha = 1$, then (1.1) reduces to the nonlinear delay dynamic equation

$$\left(r_2(t)\left[(r_1(t)x^\triangle(t))^\triangle\right]\right)^\triangle + q(t)f(x[\tau(t)]) = 0, \quad t \in \mathbb{T},\ t \ge t_0. \tag{1.4}$$

In 2005, Erbe et al. [16] considered the general third-order nonlinear dynamic equation (1.2). By employing generalized Riccati transformation techniques, they established some sufficient conditions which ensure that every solution of Eq. (1.2) is oscillatory or converges to zero. In 2007, Erbe et al. [17] studied the third-order linear dynamic equation (1.3), and they obtained Hille and Nehari type oscillation criteria for it. In 2011, Han, Li, Sun, and Zhang [18] extended and improved the results of [16,17], meanwhile obtaining some oscillatory criteria for Eq. (1.4). On this basis, we discuss the oscillation of solutions of Eq. (1.1). By using the generalized Riccati transformation and the inequality technique, we obtain some sufficient conditions which guarantee that every solution of Eq. (1.1) is oscillatory or converges to zero.

The paper is organized as follows. In Section 2, we present some basic definitions and useful results from the theory of calculus on time scales. In Section 3, we give several lemmas. In Section 4, we use the generalized Riccati transformation and the inequality technique to obtain some sufficient conditions which guarantee that every solution of Eq. (1.1) is either oscillatory or converges to zero.

2. Some preliminaries

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g:

$$(fg)^\triangle(t) = f^\triangle(t)g(t) + f(\sigma(t))g^\triangle(t) = f(t)g^\triangle(t) + f^\triangle(t)g(\sigma(t)), \tag{2.1}$$

$$\left(\frac{f}{g}\right)^\triangle(t) = \frac{f^\triangle(t)g(t) - f(t)g^\triangle(t)}{g(t)g(\sigma(t))} \quad \text{if } gg^\sigma \ne 0. \tag{2.2}$$

For $b, c \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of $f^\triangle$ is defined by

$$\int_b^c f^\triangle(t)\triangle t = f(c) - f(b).$$

The integration by parts formula reads

$$\int_b^c f^\triangle(t)g(t)\triangle t = f(c)g(c) - f(b)g(b) - \int_b^c f^\sigma(t)g^\triangle(t)\triangle t,$$

and infinite integrals are defined by

$$\int_b^\infty f(s)\triangle s = \lim_{t\to\infty}\int_b^t f(s)\triangle s.$$

For more details, see [4,5].