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Comprehensive study of intersection curves in $\mathbb{R}^4$ based on the system of ODEs

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ABSTRACT

We propose a new approach for the study of intersection curves $\mathbf{c}_I$ in $\mathbb{R}^4$ to obtain a general method for determining the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ and curvatures $\kappa_1, \kappa_2, \kappa_3$ of $\mathbf{c}_I$. Our method is based on the theory of differential equations, which distinguishes it from the classical methods used in previous works [4–6] that were based mainly on differential geometry. Moreover, our method is suitable for any form of hypersurface that defines the intersection curves in $\mathbb{R}^4$.

1. Introduction

In general, the intersection of two surfaces in $\mathbb{R}^3$ generates an intersection curve $\mathbf{c}_I$. Determining this intersection curve is an important focus of studies in CAD, CAM, CAGD, and other areas [1]. This is known as the intersection problem, or more precisely, the surface–surface intersection (SSI) problem. Many studies have been conducted on this problem and related areas, while many others are ongoing. In general, however, we cannot determine the exact parametric representation $\mathbf{c}_I(t)$ of $\mathbf{c}_I$, so we have to use appropriate approximation curves $\mathbf{a}(t)$ for practical applications [2]. The differential geometric properties of $\mathbf{c}_I$ are essential for this purpose, and these properties can be represented using a Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$, curvature κ, and torsion τ [3].

As extensions of the results for spatial intersection curves, intensive investigations of intersection curves in $\mathbb{R}^4$ were presented by Aléssio, Düldül and Abdel-All et al. [4–6] (previous research on intersection curves has been reviewed in these publications). There are four different types of intersection curves in $\mathbb{R}^4$ depending on the representation forms of three hypersurfaces that are used to generate the intersection curve. They use appropriate methods for determining the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ and curvatures $\kappa_1, \kappa_2, \kappa_3$ at a given point P on $\mathbf{c}_I$ with different types of intersection curves. In particular, Aléssio [4] provided formulas for computing the Frenet frame and curvatures using the first four derivatives of $\mathbf{c}_I$. It should be noted that all previous studies were based on the classical theory of differential geometry.

Here, we propose a new approach to investigate intersection curves $\mathbf{c}_I$ in $\mathbb{R}^4$, which is based on the theory of differential equations. We will define a system of differential equations based on the intersection equations of $\mathbf{c}_I$. We will demonstrate that the intersection equations also define an implicit curve in the domain of the intersection equations. The theory of differential equations gives us a local analytic parametrization of this implicit curve. This parametrization will be used to

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determine the first four derivatives of the intersection curve, so we can obtain the Frenet frame and the curvatures of any type of intersection curves in $\mathbb{R}^4$ using the formulas presented in [4].

During the development of this approach, we assume that all functions defining the hypersurfaces $S_k \subset \mathbb{R}^4$, $k = 1, 2, 3$ are analytic.[1] Furthermore, we assume that the given point P on $\mathbf{c}_I$ is a regular point. This means that the intersections of hypersurfaces are transversal at P, so the surface normal vectors $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ of hypersurfaces S_k, $k = 1, 2, 3$ at P are linearly independent.

The remainder of this paper is organized as follows. In the next section, we present some necessary background material. In Section 2.1, we present brief definitions of the Frenet frame and the curvatures of curves in $\mathbb{R}^4$. Formulas for the Frenet frame and curvatures, which were presented in [4], are restated in this subsection. The theory of differential equations is developed in Section 2.2. This subsection contains the definition and properties of analytic functions, the existence of a uniqueness theorem, the definition and properties of the Nambu system, and a power series method. In Section 3, we classify four types of intersection curves in $\mathbb{R}^4$ and describe previous research. In Section 4, we develop a general procedure for determining the first four derivatives of the four types of intersection curves. We also verify the analytic parametrization of the implicit curve in any dimensional space. In Section 5, we present examples that demonstrate the generality of our procedure and that justify our new approach. We conclude our study in Section 6.

2. Preliminaries

2.1. Differential geometry of a parametric curve in $\mathbb{R}^4$

We will use the following notation for a curve $\mathbf{c}$:

- ▷ $\mathbf{c}(t)$ denotes an arbitrary regular parametrization;
- ▷ $\mathbf{c}(s)$ denotes a parametrization based on the arc length;
- ▷ $\dot{\mathbf{c}} = \frac{d\mathbf{c}}{dt}, \ddot{\mathbf{c}} = \frac{d^2\mathbf{c}}{dt^2}, \ \ldots\ , \overset{(n+1)}{\mathbf{c}} = \frac{d^{n+1}\mathbf{c}}{dt^{n+1}}$;
- ▷ $\mathbf{c}' = \frac{d\mathbf{c}}{ds}, \mathbf{c}'' = \frac{d^2\mathbf{c}}{ds^2}, \ \ldots\ , \mathbf{c}^{(n+1)} = \frac{d^{n+1}\mathbf{c}}{ds^{n+1}}$.

Definition 1. Let $\mathbf{c}(s)$ be a parametric curve with the arc length parameter s, while $\mathbf{t} = \mathbf{c}'(s)$ is the unit tangent vector of $\mathbf{c}(s)$ in $\mathbb{R}^4$. We define the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}_1, \mathbf{b}_2\}$ and the first, second, and third curvatures as follows:

$$\begin{pmatrix} \mathbf{t}'(s) \\ \mathbf{n}'(s) \\ \mathbf{b}_1'(s) \\ \mathbf{b}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}_1(s) \\ \mathbf{b}_2(s) \end{pmatrix}.$$

Definition 2. The cross-product $\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3$ of three vectors in $\mathbb{R}^4$ is defined as

$$\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3 = \det \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ v_1^1 & v_1^2 & v_1^3 & v_1^4 \\ v_2^1 & v_2^2 & v_2^3 & v_2^4 \\ v_3^1 & v_3^2 & v_3^3 & v_3^4 \end{pmatrix},$$

where $\mathbf{v}_i = (v_i^1, v_i^2, v_i^3, v_i^4)$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ are the coordinate vectors.

Next, we restate Theorems 4 and 5 from [4].

Theorem 3. *For any regular C^4 parametric curve $\mathbf{c}(t)$ in $\mathbb{R}^4$, we have*

$$\begin{aligned} \mathbf{b}_2(t) &= \frac{\dot{\mathbf{c}}(t) \otimes \ddot{\mathbf{c}}(t) \otimes \dddot{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t) \otimes \ddot{\mathbf{c}}(t) \otimes \dddot{\mathbf{c}}(t)\|}, \qquad \mathbf{b}_1(t) = \frac{\dot{\mathbf{c}}(t) \otimes \ddot{\mathbf{c}}(t) \otimes \mathbf{b}_2(t)}{\|\dot{\mathbf{c}}(t) \otimes \ddot{\mathbf{c}}(t) \otimes \mathbf{b}_2(t)\|}, \\ \mathbf{n}(t) &= \frac{\dot{\mathbf{c}}(t) \otimes \mathbf{b}_1(t) \otimes \mathbf{b}_2(t)}{\|\dot{\mathbf{c}}(t) \otimes \mathbf{b}_1(t) \otimes \mathbf{b}_2(t)\|}, \qquad \mathbf{t}(t) = \frac{\dot{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t)\|}. \end{aligned} \tag{1}$$

Theorem 4. *For any regular C^4 parametric curve $\mathbf{c}(t)$ in $\mathbb{R}^4$, we have*

$$\kappa_1(t) = \frac{\mathbf{n}(t) \cdot \ddot{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t)\|^2}, \qquad \kappa_2(t) = \frac{\mathbf{b}_1(t) \cdot \dddot{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t)\|^3 \kappa_1(t)}, \qquad \kappa_3(t) = \frac{\mathbf{b}_2(t) \cdot \overset{....}{\mathbf{c}}(t)}{\|\dot{\mathbf{c}}(t)\|^4 \kappa_1(t) \kappa_2(t)}. \tag{2}$$

[1] A function is said to be analytic if it has a local power series expression. Multivariate polynomials are examples of analytic functions [7].