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On pricing barrier options with regime switching

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ABSTRACT

We consider the valuation of both European-style and American-style barrier options in a Markovian, regime-switching, Black–Scholes–Merton economy, where the price process of an underlying risky asset is governed by a Markovian, regime-switching, geometric Brownian motion. Both the probabilistic and partial differential equation (PDE), approaches are used to price the barrier options. For the probabilistic approach to value a European-style barrier option, we employ the fundamental matrix solution and the Fourier transform space to derive a (semi)-analytical solution. The PDE approach is employed to value an American barrier option, where we obtain a system of free-boundary, coupled PDEs and an analytical quadratic approximation to the price by solving the free-boundary problem.

1. Introduction

Barrier options are one of the major types of options actively used in both exchange-traded and over-the-counter markets. They are derivative securities that are either nullified, (''out''), or in force, (''in''), when the price process of an underlying risky asset breaches a pre-specified level, namely, a barrier level. They can be further classified as down-and-out, down-and-in, up-and-out and up-and-in, depending on the initial price of the risky asset in relation to the barrier level. Like other derivative securities, barrier options can be used for risk management. In particular, they provide an insurance protection against the risk triggered by the event that the price of the risky asset crosses above, (or below), a particular level. Barrier options have important applications in corporate finance. In particular, they can be used as an endogenous default for valuing corporate debts and other defaultable securities.

Given the significance role played by barrier options, the valuation of barrier options is an important issue in the theory and practice of finance. In the standard Black–Scholes–Merton model, (i.e., geometric Brownian motion (GBM) asset price process), closed-form pricing formulas for European-style barrier options of various types were obtained, (see, for example, [1–3]). The valuation of barrier options has been considered in the finance literature for more realistic models for the price of the underlying risky asset. However, even for the case of a GBM with time-dependent coefficients, the valuation of barrier options is not a trivial issue. Nevertheless, some approximation methods have been proposed. Roberts and Shortland [4] adopted the hazard-rate tangent approximation to evaluate both the lower and upper bounds for the price of a European-style barrier option in a time-dependent GBM. Lo et al. [5] provided an accurate closed-form estimate for the price of a European-style barrier option in the time-dependent GBM. For the case of Lèvy processes, explicit

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solutions were obtained for European-style barrier options. Kou and Wang [6] obtained a closed-form solution to the Laplace transform for a European-style barrier option when the share price dynamics follow a double exponential jump–diffusion process. Boyarchenko and Levendorskiĭ [7] adopted the Wiener–Hopf factorization to derive an analytical pricing formula for European-style barrier options in the case of an exponential Lévy process.

Besides considering more realistic models for the price of the underlying risky asset, the valuation of barrier options with more delicate contractual structures has also been considered. Kunitomo and Ikeda [8] adopted a probabilistic approach to value European-style double barrier options. The valuation of American-style barrier options is a complicated issue. Closed-form pricing formulas for American-style barrier options do not exist, except for the simple cases of American down-and-out and down-and-in calls without paying dividends (see, for example, [9]). Some numerical methods have been employed to price American-style barrier options. For example, lattice methods were adopted by Boyle and Lau [10], Ritchken [11], Cheuk and Vorst [12] and Figlewski and Gao [13]. Finite difference methods were employed by Boyle and Tian [14] and Zvan et al. [15]. Analytical approximation methods were developed by Gao et al. [16] and Zhang and Taksar [17].

Recently there has been a considerable interest in the application of regime-switching models to investigate option valuation problems. Some examples include [18–21]. The basic idea of regime-switching models is to allow the model parameters to change over time according to a state process, which is usually modeled as a Markov chain. A key advantage of regime-switching models is to incorporate the impact of structural changes in economic conditions on the price dynamics. Regime-switching models can also incorporate other important stylized facts on asset returns such as asymmetry and heavy-tailedness in the distribution of asset returns. Due to the presence of an additional source of uncertainty described by the Markov chain, the market described by regime-switching models is, in general, incomplete. Consequently, there is more than one equivalent martingale measure for valuation. Different methods have been introduced in the literature. Guo [19] introduced the idea of completing a regime-switching market by a set of "fictitious" securities corresponding to different states of a Markov chain. Then the pricing of options was done in the completed market. Elliott, Chan and Siu [21] proposed the use of the regime-switching Esscher transform to select an equivalent martingale measure in the incomplete market and justified the use of the Esscher transform as the minimal entropy martingale measure.

In this paper, we consider the valuation of both European-style and American-style barrier options in a Markovian, regime-switching, Black–Scholes–Merton economy, where the price process of an underlying risky asset is governed by a Markovian, regime-switching, geometric Brownian motion. The key model parameters such as the market interest rate of a bond, the appreciation rate and the volatility of a share are modulated by a continuous-time, finite-state, Markov chain. The states of the chain represent different states of an economy. Here we first employ the regime-switching Esscher transform to select an equivalent martingale measure, or a pricing kernel, for valuation in the incomplete market described by the regime-switching asset price model. Both the probabilistic and partial differential equation (PDE), approaches are used to price the barrier options. For the probabilistic approach to value a European-style barrier option, we employ the fundamental matrix solution and the Fourier transform space to derive a (semi)-analytical solution. We adopt the PDE approach to price an American-style barrier option, where we obtain a system of free-boundary, coupled PDEs and an analytical quadratic approximation to the price by solving the free-boundary problem.

This paper is organized as follows. The next section presents the model dynamics and the use of the regime-switching Esscher transform to select a pricing kernel. Section 3 discusses the valuation of European-style barrier options. Section 4 derives an analytical quadratic approximation for the price of American-style up-and-out put options in a two-state regime switching model. Section 5 presents an algorithm for determining the approximate price. The final section gives concluding remarks.

2. The model dynamics and the valuation

This section aims at presenting the model dynamics and the valuation method for a European-style barrier option in a Markovian regime-switching Black–Scholes–Merton economy. A regime-switching version of the Esscher transform in [21] is employed to specify a pricing kernel and the price dynamics of the risky share under an equivalent martingale measure selected by the regime-switching Esscher transform are presented. Furthermore, a conditional price of a European-style up-and-out put option is given.

We consider a continuous-time financial model with two primitive securities, a bond B and a share S. These securities can be traded continuously over time in a finite time horizon $\mathcal{T} := [0, T]$, where $T < \infty$. To model uncertainty, we consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability measure.

To describe the evolution of the state of an economy over time, we introduce a continuous-time, finite-state, Markov chain $\mathbf{X} := \{\mathbf{X}(t) | t \in \mathcal{T}\}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ with state space $\mathcal{E} := \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_N\} \subset \Re^N$, where the jth-component of $\mathbf{e}_i$ is the Kronecker product δ_{ij}, for each $i, j = 1, 2, \dots, N$. Here we follow the convention of Elliott et al. [22] and identify the state space of the chain $\mathbf{X}$ by the canonical state space $\mathcal{E}$. We suppose that the chain $\mathbf{X}$ is observable. The states of the chain $\mathbf{X}$ can be interpreted as proxies of different levels of observable (macro)-economic indicators, such as the gross domestic product or the retail price index. They may also be interpreted as credit ratings of a corporation.

Let $\mathbf{A} := [a_{ji}]_{i,j=1,2,\dots,N}$ be the rate matrix of the chain $\mathbf{X}$, where a_{ji} is the constant rate of transition of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$. The statistical properties of the chain $\mathbf{X}$ are described by the rate matrix $\mathbf{A}$. With the canonical state space