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On concomitants of upper record statistics and survival analysis for a pseudo-Gompertz distribution

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ABSTRACT

This paper presents upper record statistics and their concomitants for a bivariate pseudo-Gompertz distribution about paired lifetime variables. Survival and hazard functions are derived for the distribution. The survival and hazard functions are displayed for some selected values of the parameters of concern. Interpretations are given for the potential reliability and actuarial applications of the obtained results.

1. Introduction

Records on a random variable X are realized as a sequence of observations that are larger or smaller than all the previously observed ones in the sequence. So, the record values are distinct elements in the successive maxima or minima about a sequence of a random variable. Every change in the maximum during the observation process for X means that a record is observed. In this regard, records are dealt with in the scope of the extremal processes as discussed in [1]. Many theoretical or applied areas of science use the theory and methodology about the record values in the empirical investigations about the timing and magnitude of record type extremities.

Let $\{(X_i, Y_i), i = 1, \ldots, n\}$ be a random sample from a bivariate distribution function $F(x, y)$ of a random vector (X, Y). The values in the random sample are observed as the realizations of the $\{(X_i, Y_i), i = 1, 2, \ldots\}$ sequence of identically distributed bivariate random variables with a common joint distribution function $F(x, y)$. The $\{X_i\}$ sequence from $\{(X_i, Y_i), i = 1, 2, \ldots\}$ is itself a sequence of random variables with a common distribution function F_X which is the marginal distribution of X from the joint distribution function $F(x, y)$. Further, T indicates the time of observations on (X, Y) such that $T_1 = 1$ and $T_r = \inf\left\{k : k > T_{r-1}, X_k > X_{T_{r-1}}\right\}$. Letting $R_r = X_{T_r}$, the sequence $\{R_r\}$ is defined as the sequence of the upper records of $\{X_i\}$, and the corresponding Y-coordinates Y_{T_r}, denoted by $R_{[r]}$, are defined as the sequence of concomitants $\left\{R_{[r]}\right\}$ of the upper records. Lower record values and their concomitants are defined similarly as seen in [1–3].

The statistical features and potential applications of record values were introduced in the early fifties by Candler [2]. Since then, a rich literature has grown on the statistical theory and methodology for the data analysis and inference about the record values as seen in [1,3–6].

This paper undertakes the random vector (X, Y) to represent paired lifetimes of components of a physical system or lifetimes of human beings in a population. The coupling of Y with X can be expressed by some analytical functions in order

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to associate Y with X up to the investigation and modeling interests of the researchers. Several probability distribution models can be considered for the random lifetimes vector (X, Y). Life distribution models are discussed in depth by Marshall and Olkin [7]. Among the known life distribution models, the Gompertz distribution attracts attention as a suitable model to compute age specific lifetime probabilities and to express hazard rates for system components or mortality rates for individuals in flexible and meaningful ways [8,9]. We consider the bivariate Gompertz distribution for the survival modeling attempts about the paired lifetimes and propose a bivariate pseudo-Gompertz distribution along the lines of pseudo-distributions. A relatively recent work by Filus and Filus [10] presents the definition and theoretical background of the pseudo-distributions.

The following sections of the paper begins with the definitions and distributions of the upper record values and their concomitants. Then, a bivariate pseudo-Gompertz distribution is introduced, and the distributional properties of the upper record values and their concomitants are derived for the new distribution. Thereafter, the survival analysis models are presented and some interpretations and implications about these models are provided. And, a conclusion is given in the sequel.

2. Concomitants of upper record statistics

The theory and applications for the upper record values and their concomitants have been enriched by the recent works of [11,12]. Special functions about the record statistics and concomitants are presented in [13] for the applied scientists. Following them, we denote the upper record statistics and their concomitants, as already written in the first section, with $\left(R_r, R_{[r]}\right) = \left(X_{T_r}, Y_{T_r}\right)$, such that $T_r = \inf\left\{k : k > T_{r-1}, X_k > X_{T_{r-1}}\right\}$. So, it is obvious that $T_1 = 1$ and $\left(R_1, R_{[1]}\right) = (X_1, Y_1)$ with probability one.

Recall that $\{(X_i, Y_i), i = 1, 2, \ldots\}$ is a sequence of identically and independently distributed bivariate random variables with a common distribution $F(x, y)$, and $\{X_i\}$ is a sequence of identically and independently distributed random variables with the common marginal distribution function $F(x)$. Let $f(x)$ denote the marginal probability density function of X. As one can see in [3–5], the probability density function of the r-th upper record R_r is defined as

$$f_{R_r}(x) = \frac{1}{\Gamma(r)} f(x) \left[H(x)\right]^{r-1} \tag{1}$$

where $H(x) = -1n\left[1 - F(x)\right]$. The joint probability density function of the r-th and s-th upper records R_r and R_s, $r < s$, is presented by Ahsanullah [3] using the following general expression

$$f_{R_r,R_s}(x_1, x_2) = \frac{h(x_1) f(x_2)}{\Gamma(r)\Gamma(s-r)} \left[H(x_1)\right]^{r-1} \left[H(x_2) - H(x_1)\right]^{s-r-1} \tag{2}$$

where $h(x_r) = \frac{d}{dx_r} H(x_r)$ and $-\infty < x_1 < x_2 < \infty$.

The probability density function for the concomitant of the r-th upper record is also given by Ahsanullah [3] as:

$$f_{R_{[r]}}(y) = \int_{-\infty}^{\infty} f(y \mid x) f_{R_r}(x) dx. \tag{3}$$

The joint probability density function of the concomitants of the r-th and s-th upper record values can be computed, as shown by Ahsanullah [3], in the following form

$$f_{R_{[r]},R_{[s]}}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{x_s} f(y_1 \mid x_1) f(y_2 \mid x_2) f_{R_r,R_s}(x_1, x_2) dx_1 dx_2 \tag{4}$$

where $f_{R_r,R_s}(x_1, x_2)$ is expressed in (2).

Under the assumptions of the model, the expressions given above hold for any bivariate distribution $F(x, y)$.

3. Concomitants of record values for bivariate pseudo-Gompertz distribution

A pseudo-distribution is a function that contains linear combinations of the underlying random variables in its parameters and satisfies all properties that are required to be a probability distribution function. A new class of pseudo-distributions is introduced in [10,14] for statistical applications where an actual distribution cannot be used easily. A multivariate pseudo-distribution is presented by Diaz-Garcia et al. [15]. Some new and specific pseudo-distributions and concomitants for them have been proposed and discussed by Shahbaz and Ahmad [16] and Shahbaz et al. [17]. Along similar lines, the ongoing research by the authors of this paper has produced a new bivariate pseudo-Gompertz distribution for a random vector (X, Y) that associates the random variable Y with X through the real valued function $\phi(x) = e^{\mu_1 x} - 1$ with $\mu_1 > 0$ and $x > 0$.

Using the marginal Gompertz density functions for X and Y

$$f(x; \lambda, \mu_1) = \lambda e^{\mu_1 x} \exp\left[-\frac{\lambda}{\mu_1}\left(e^{\mu_1 x} - 1\right)\right], \quad \mu_1 > 0, \lambda > 0, x > 0,$$

$$f(y; \phi(x), \mu_2 | x) = \phi(x) e^{\mu_2 y} \exp\left[-\frac{\phi(x)}{\mu_2}\left(e^{\mu_2 y} - 1\right)\right], \quad \mu_2 > 0, x > 0, y > 0,$$