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Newton's method for solving a class of nonlinear matrix equations☆

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ABSTRACT

In this paper, a class of nonlinear matrix equations $X = Q + A^H(I \otimes X - C)^{-1}A$ is considered. An equivalent form of this equation is derived, and the properties of the solutions are studied. Then Newton's iterative method is applied to solve this equivalent one. Finally, two numerical examples are given to illustrate the efficiency of Newton's method.

1. Introduction

In this paper, we consider the matrix equation

$$X = Q + A^H(I \otimes X - C)^{-1}A, \tag{1}$$

where Q is an $n \times n$ Hermitian positive definite matrix, C is an $mn \times mn$ Hermitian positive semi-definite matrix, A is an $mn \times n$ complex matrix, and I is the identity matrix of order m.

This matrix equation is closely related to a certain interpolation problem [1–3]. If $m = 1$ and $C = 0$, Eq. (1) becomes

$$X - A^H X^{-1} A = Q,$$

which arises from the analysis of stationary Gaussian reciprocal processes over a finite interval [4]. This class of matrix equations has been studied by several authors [4–14].

The following notations are used throughout this paper. Let $\mathbb{R}^+$ denote the set of positive real numbers. For matrices A and B, let A^H denote the conjugate transpose of the matrix A, $A \otimes B$ denote the Kronecker product of A and B, and $\|A\|$ denote the spectral norm of A. We also use $A > 0$ ($A \geq 0$) to indicate that A is a positive definite (semi-definite) matrix and $A > B$ ($A \geq B$) to stand for $A - B > 0$ ($A - B \geq 0$).

Let $P(n)$ denote the set of $n \times n$ positive definite matrices. From [1] we know that under the condition $I \otimes Q > C$, Eq. (1) has a unique positive definite solution in the set $S(n) = \{X \in P(n) | I \otimes X > C\}$, and the sequence generated by the iteration

$$X_{k+1} = Q + A^H(I \otimes X_k - C)^{-1}A \tag{2}$$

converges to this unique solution for $X_0 \in S(n)$.

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In this paper, we will derive an equivalent form of Eq. (1), and study the properties of the solutions. Then we will discuss Newton's iterative method for solving the equivalent equation. Finally, we will give two examples to illustrate the efficiency of Newton's iterative method.

Hereafter we will assume that $I \otimes Q - C > 0$.

2. On the equivalent form of Eq. (1)

In this section, we will derive an equivalent equation of Eq. (1), and discuss the properties of the solutions of this equivalent one under certain conditions.

Lemma 2.1. *Eq. (1) is equivalent to nonlinear matrix equation*

$$Y = I + B^H (I \otimes Y^{-1})B, \tag{3}$$

where

$$Y = (I \otimes Q - C)^{-1/2}(I \otimes X - C)(I \otimes Q - C)^{-1/2},$$
$$B = [I \otimes (I \otimes Q - C)^{-1/2}](I \otimes A)(I \otimes Q - C)^{-1/2},$$

that is, Eq. (3) has a positive definite solution

$$\tilde{Y} = (I \otimes Q - C)^{-1/2}(I \otimes \tilde{X} - C)(I \otimes Q - C)^{-1/2} \in P(mn) \tag{4}$$

provided that $\tilde{X}$ is a solution of Eq. (1) in $S(n)$; conversely,

$$\tilde{X} = Q + A^H (I \otimes Q - C)^{-1/2}(\tilde{Y}^{-1})(I \otimes Q - C)^{-1/2}A \tag{5}$$

is a solution of Eq. (1) in $S(n)$ if $\tilde{Y}$ is a positive definite solution of Eq. (3).

Proof. Suppose $\tilde{X}$ is a solution of Eq. (1) in $S(n)$, then $I \otimes \tilde{X} - C > 0$. Let

$$\tilde{Y} = (I \otimes Q - C)^{-1/2}(I \otimes \tilde{X} - C)(I \otimes Q - C)^{-1/2}.$$

Then $\tilde{Y}$ is positive definite since $I \otimes Q - C > 0$. Notice that

$$\begin{aligned} B^H (I \otimes \tilde{Y}^{-1})B &= (I \otimes Q - C)^{-1/2}[I \otimes (A^H (I \otimes \tilde{X} - C)^{-1}A)](I \otimes Q - C)^{-1/2}. \\ &= (I \otimes Q - C)^{-1/2}[I \otimes (\tilde{X} - Q)](I \otimes Q - C)^{-1/2} \\ &= (I \otimes Q - C)^{-1/2}[I \otimes \tilde{X} - C](I \otimes Q - C)^{-1/2} - (I \otimes Q - C)^{-1/2}[I \otimes Q - C](I \otimes Q - C)^{-1/2} \\ &= \tilde{Y} - I, \end{aligned}$$

so $\tilde{Y}$ is a positive definite solution of Eq. (3).

Conversely, suppose that Eq. (3) has a positive definite solution $\tilde{Y}$. Let

$$\tilde{X} = Q + A^H (I \otimes Q - C)^{-1/2}(\tilde{Y}^{-1})(I \otimes Q - C)^{-1/2}A.$$

Since

$$\begin{aligned} I \otimes \tilde{X} - C &= I \otimes Q + (I \times A^H)[I \otimes (I \otimes Q - C)^{-1/2}](I \otimes \tilde{Y}^{-1})[I \otimes (I \otimes Q - C)^{-1/2}](I \times A) - C \\ &= (I \otimes Q - C)^{1/2}[I + (I \otimes Q - C)^{-1/2}(I \times A^H)[I \otimes (I \otimes Q - C)^{-1/2}](I \otimes \tilde{Y}^{-1}) \\ &\quad \cdot [I \otimes (I \otimes Q - C)^{-1/2}](I \times A)(I \otimes Q - C)^{-1/2}](I \otimes Q - C)^{1/2} \\ &= (I \otimes Q - C)^{1/2}(I + B^H (I \otimes \tilde{Y}^{-1})B)(I \otimes Q - C)^{1/2}, \\ &= (I \otimes Q - C)^{1/2}\tilde{Y}(I \otimes Q - C)^{1/2}, \end{aligned}$$

we have

$$Q + A^H (I \otimes \tilde{X} - C)^{-1}A = Q + A^H (I \otimes Q - C)^{-1/2}\tilde{Y}^{-1}(I \otimes Q - C)^{-1/2}A = \tilde{X},$$

thus, $\tilde{X}$ is a solution of Eq. (1). Obviously, $\tilde{X} > Q$, so $I \otimes \tilde{X} > I \otimes Q > C$, that is, $\tilde{X} \in S(n)$. □

According to Lemma 2.1, we only need to discuss the properties of Eq. (3).

Hereafter, we will denote $B = (B_1^T \cdots B_m^T)^T$, $B_i \in \mathbb{C}^{mn}$, $i = 1, \ldots, m$.