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Mean square stability and dissipativity of two classes of theta methods for systems of stochastic delay differential equations☆

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ABSTRACT

In this paper, we first study the mean square stability of numerical methods for stochastic delay differential equations under a coupled condition on the drift and diffusion coefficients. This condition admits that the diffusion coefficient can be highly nonlinear, i.e., it does not necessarily satisfy a linear growth or global Lipschitz condition. It is proved that, for all positive stepsizes, the classical stochastic theta method with $\theta \geq 0.5$ is asymptotically mean square stable and the split-step theta method with $\theta > 0.5$ is exponentially mean square stable. Conditional stability results for the methods with $\theta < 0.5$ are also obtained under a stronger assumption. Finally, we further investigate the mean square dissipativity of the split-step theta method with $\theta > 0.5$ and prove that the method possesses a bounded absorbing set in mean square independent of initial data.

1. Introduction

Stochastic delay differential equations (SDDEs) play an important role in many applications (cf. [1,2]). In recent years, the numerical analysis of such equations has attracted a lot of attention. For example, Küchler and Platen [3] discussed strong discrete time approximation of SDDEs. Baker and Buckwar [4] studied the convergence of explicit one-step methods. Mao and Sabanis [5] investigated the convergence of the Euler–Maruyama method under the local Lipschitz condition. Hu et al. [6] introduced the Milstein scheme for SDDEs.

One of the interesting problems in numerical analysis is the investigation of the stability of numerical methods. For example, Liu et al. [7] studied the mean-square stability of the stochastic theta method for linear scalar model equations. Baker and Buckwar [8] analyzed the exponential stability in p-th mean of the stochastic theta method by using Halanay inequality. In recent years, nonlinear stability of numerical methods has also received attention. Mao [9] proved that when the stepsize is sufficiently small, the Euler–Maruyama method can reproduce the mean-square exponential stability of the underlying SDDEs which satisfy the global Lipschitz condition. Wang and Zhang [10] obtained some stability conditions for the Milstein method. Wu et al. [11] studied the almost sure exponential stability of numerical methods. Wang and Gan [12] investigated the mean-square exponential stability of a split-step Euler method. Huang et al. [13,14] studied the delay-dependent stability of the stochastic theta method. To the best of our knowledge, however, all above results are derived for SDDEs of which the diffusion coefficient satisfies a linear growth or global Lipschitz condition.

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In this paper, we study the stability of numerical methods under a coupled condition on the drift and diffusion coefficients. This condition admits that the diffusion coefficient is highly nonlinear, i.e., it does not necessarily satisfy the linear growth or global Lipschitz condition. The classical stochastic theta method and the split-step theta method are considered and some mean-square stability results are obtained. This paper is organized as follows. In Section 2, the two classes of theta methods are recalled. In Section 3, some exponential stability and asymptotic stability results are derived. In Section 4, we further study the mean square dissipativity of the split-step theta method. Finally, in Section 5, we give some numerical experiments.

2. Two classes of theta methods for SDDEs

Let $\{\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e., it is increasing and right continuous, and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t) = (w^1(t), \dots, w^l(t))^T$ be standard l-dimensional Brownian motion defined on the probability space. Let $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times l}$ be given mappings, where $\mathbb{R}_+ = [0, +\infty)$. Consider d-dimensional Itô SDDEs of the form

$$\begin{cases} dy(t) = f(t, y(t), y(t-\tau))dt + g(t, y(t), y(t-\tau))dw(t), \quad t \geq 0, \\ y(t) = \phi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where the delay τ is a positive constant, and $\phi(t)$ is an $\mathscr{F}_0$-measurable, $C([-\tau, 0]; \mathbb{R}^d)$-valued random variable satisfying

$$\sup_{-\tau\leq t\leq 0} \mathbb{E}[\phi^T(t)\phi(t)] < +\infty, \tag{2.2}$$

with the notation $\mathbb{E}$ denoting the mathematical expectation with respect to $\mathbb{P}$.

There exist many numerical schemes for stochastic ordinary differential equations in the literature (see, e.g., [15–18]). If an appropriate interpolation procedure for the delay argument is employed, these schemes can be adapted to solve SDDEs. An adaptation of the classic stochastic theta method to (2.1) leads to

$$y_{n+1} = y_n + \theta\Delta t f(t_{n+1}, y_{n+1}, \bar{y}_{n+1}) + (1-\theta)\Delta t f(t_n, y_n, \bar{y}_n) + g(t_n, y_n, \bar{y}_n)\Delta w_n, \tag{2.3}$$

where $\Delta t > 0$ is the time stepsize, $t_n = n\Delta t$, y_n is an approximation to $y(t_n)$, $\theta \in [0, 1]$ is a fixed parameter, $\Delta w_n = w(t_{n+1}) - w(t_n)$, and $\bar{y}_n$ denotes an approximation to the delay argument $y(t_n - \tau)$.

For an arbitrarily fixed time stepsize Δt, there exist a unique positive integer m and a real number $\delta \in [0, 1)$ such that $\tau = (m-\delta)\Delta t$. This implies that $y(t_n - \tau) = y(t_{n-m} + \delta\Delta t)$. Therefore, it is natural to define $\bar{y}_n$ by the linear interpolation

$$\bar{y}_n = \delta y_{n-m+1} + (1-\delta)y_{n-m}, \tag{2.4}$$

where $y_n = \phi(t_n)$ for $n \leq 0$.

In order to distinguish this method and another method with parameter θ below, we will refer to (2.3) as the stochastic linear theta (SLT) method following the notations in [19].

An adaptation of the split-step theta (SST) method in [19] to problem (2.1) leads to

$$Y_n = y_n + \theta\Delta t f(t_n + \theta\Delta t, Y_n, \bar{Y}_n), \tag{2.5}$$

$$\bar{Y}_n = \delta Y_{n-m+1} + (1-\delta)Y_{n-m}, \tag{2.6}$$

$$y_{n+1} = y_n + \Delta t f(t_n + \theta\Delta t, Y_n, \bar{Y}_n) + g(t_n + \theta\Delta t, Y_n, \bar{Y}_n)\Delta w_n, \tag{2.7}$$

where $Y_n = \phi(t_n + \theta\Delta t)$ for $n < 0$. Here we use the equi-stage linear interpolation technique [20] to approximate the delay argument. In the case of deterministic delay equations (i.e., $g = 0$), it is known that this interpolation can lead to desirable linear and nonlinear stability properties (cf. [20,21]). We naturally expect that it has a good performance for stochastic equations.

In the special case of $\theta = 1$, this method is equivalent to the split-step backward Euler method, which was first proposed for stochastic ordinary differential equations in [22] and which has been applied to SDDEs in [12]. We also mention that there exist some other types of split methods with parameter θ in the literature (see, e.g., [23–25]). The reason why we consider scheme (2.5)–(2.7) is that we can establish some provable stability results for it. In particular, it possesses a better exponential mean square stability property than the classic SLT method.

Now we recall some stability concepts for numerical methods.

Definition 2.1. For a given stepsize Δt, a numerical method is said to be exponentially stable in mean square if there is a pair of positive constants γ and C such that for any initial data $\phi(t)$ the numerical solution y_n produced by the method satisfies

$$\mathbb{E}[y_n^T y_n] \leq Ce^{-\gamma t_n} \sup_{-\tau\leq t\leq 0} \mathbb{E}[\phi^T(t)\phi(t)], \quad \forall n \geq 0.$$