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Numerical approximation of a two-dimensional parabolic time-dependent problem containing a delta function[☆]

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ABSTRACT

The convergence of a difference scheme for a two-dimensional initial-boundary value problem for the heat equation with concentrated capacity and time-dependent coefficients of the space derivatives is considered. An estimate of the rate of convergence in a special discrete $\widetilde{W}_2^{2,1}$ Sobolev norm, compatible with the smoothness of the coefficients and the solution, is proved.

1. Introduction

The finite-difference method is one of the basic tools for the numerical solution of partial differential equations. In the case of problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.), the solution has weak global regularity, and it is impossible to establish convergence of finite-difference schemes using the classical Taylor series expansion. Often, the Bramble–Hilbert lemma takes the role of the Taylor formula for functions from the Sobolev spaces [1–3].

Following Lazarov et al. [3], a convergence rate estimate of the form

$$\|u - v\|_{W_{2,h}^k} \le C h^{s-k} \|u\|_{W_2^s}, \quad s > k,$$

is called *compatible* with the smoothness (regularity) of the solution u of the boundary value problem. Here, v is the solution of the discrete problem, h is the spatial mesh step, W_2^s and $W_{2,h}^k$ are Sobolev spaces of functions with continuous and discrete argument, respectively, and C is a constant which does not depend on u and h. For parabolic problems, typical estimates are of the form

$$\|u - v\|_{W_{2,h\tau}^{k,k/2}} \le C(h + \sqrt{\tau})^{s-k} \|u\|_{W_2^{s,s/2}}, \quad s > k,$$

where τ is the time step. In the case of equations with variable coefficients, the constant C in the error bounds depends on the norms of the coefficients (see, for example, [2,4,5]).

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One interesting class of parabolic problems models processes in heat-conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or, equivalently, the jump of the heat flow in the singular point is proportional to the time derivative of the temperature [6]. Such problems are nonstandard, and classical tools of the theory of finite-difference schemes are difficult to apply to their convergence analysis.

In the present paper, a finite-difference scheme, approximating the two-dimensional initial-boundary value problem for the heat equation with concentrated capacity and time-dependent coefficients is derived. A special Sobolev norm (corresponding to the norm $W_2^{2,1}$ for a classical heat-conduction problem) is constructed. In this norm, a convergence rate estimate, compatible with the smoothness of the solution of the boundary value problem, is obtained.

Note that the convergence to classical solutions is studied in [7,8]. A one-dimensional parabolic problem with weak solution is studied in [9–12], and a two-dimensional parabolic problem with variable coefficients (that are independent of time) is considered in [13,12].

2. Preliminary results

Let H be a real separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, and let S be an unbounded selfadjoint positive definite linear operator, with domain $D(S)$ dense in H. It is easy to see that the product $(u,v)_S = (Su,v)$ $(u,v\in D(S))$ satisfies the axioms of an inner product. The closure of $D(S)$ in the norm $\|u\|_S = (u,u)_S^{1/2}$ is a Hilbert space $H_S \subset H$. The inner product (u,v) continuously extends to $H_S^* \times H_S$, where $H_S^* = H_{S^{-1}}$ is the dual space for H_S. The spaces H_S, H and $H_{S^{-1}}$ form a Gelfand triple $H_S \subset H \subset H_{S^{-1}}$, with continuous imbeddings. The operator S extends to the map $S : H_S \to H_S^*$. There exists an unbounded selfadjoint positive definite linear operator $S^{1/2}$, such that $D(S^{1/2}) = H_S$ and $(u,v)_S = (Su,v) = (S^{1/2}u, S^{1/2}v)$. We also define the Sobolev spaces $W_2^s(a,b;H)$ and $W_2^0(a,b;H) = L_2(a,b;H)$ of the functions $u = u(t)$ mapping the interval $(a,b)\subset R$ into H (see [14,15]).

Let A and B be unbounded selfadjoint positive definite linear operators, $A = A(t)$, $B \neq B(t)$, in the Hilbert space H, in general noncommuting, with $D(A)$ dense in H and $H_A \subset H_B$. We consider the following abstract Cauchy problem (see [16,15]):

$$B\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f(t), \quad 0<t<T; \qquad u(0)=u_0, \tag{1}$$

where $f(t)$ and u_0 are given, and $u(t)$ is an unknown function with values in H. Let us also assume that $A_0 \le A(t) \le cA_0$, where $c = \text{const} > 1$, and A_0 is a constant selfadjoint positive definite linear operator in H. We also assume that $A(t)$ is a nonincreasing operator in the variable t:

$$\left(\frac{\mathrm{d}A(t)}{\mathrm{d}t}u, u\right) \le 0, \quad \forall u \in H. \tag{2}$$

The following proposition holds.

Lemma 1. *The solution of problem (1) satisfies the a priori estimate*

$$\int_0^T \left(\|Au(t)\|_{B^{-1}}^2 + \left\|\frac{\mathrm{d}u(t)}{\mathrm{d}t}\right\|_B^2\right)\mathrm{d}t \le C\left(\|u_0\|_{A_0}^2 + \int_0^T \|f(t)\|_{B^{-1}}^2\,\mathrm{d}t\right), \tag{3}$$

provided that $u_0 \in H_{A_0}$ and $f \in L_2(0,T;H_{B^{-1}})$.

Proof. It follows from the theory of abstract parabolic initial value problems that, for $u_0 \in H_B$ and $f \in L_2(0,T;H_{A_0^{-1}})$, problem (1) has a unique solution $u \in L_2(0,T;H_{A_0})$ with $\mathrm{d}u/\mathrm{d}t \in L_2(0,T;H_{BA_0^{-1}B})$. We take the inner product of (1) with $2\mathrm{d}u/\mathrm{d}t$, and estimate the right-hand side by the Cauchy–Schwarz inequality:

$$2\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_B^2 + 2\left(Au, \frac{\mathrm{d}u}{\mathrm{d}t}\right) = 2\left(f, \frac{\mathrm{d}u}{\mathrm{d}t}\right) \le \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_B^2 + \|f\|_{B^{-1}}^2.$$

By (2), this implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_A^2) \le 2\left(Au, \frac{\mathrm{d}u}{\mathrm{d}t}\right), \quad \forall u \in H.$$

Further, we have

$$2\left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_B^2 + \frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_A^2) \le \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_B^2 + \|f\|_{B^{-1}}^2.$$

Integration with respect to t gives

$$\int_0^T \left\|\frac{\mathrm{d}u}{\mathrm{d}t}\right\|_B^2 \mathrm{d}t + \|u(T)\|_{A(T)}^2 \le \|u_0\|_{A(0)}^2 + \int_0^T \|f(t)\|_{B^{-1}}^2\,\mathrm{d}t. \tag{4}$$