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Approximation by Chlodowsky type Jakimovski–Leviatan operators

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ABSTRACT

We introduce a generalization of the Jakimovski–Leviatan operators constructed by *A. Jakimovski and D. Leviatan* (1969) in [1] *and* the theorems on convergence and the degree of convergence are established. We also give a Voronovskaya-type theorem. Furthermore, we study the convergence of these operators in a weighted space of functions on a positive semi-axis and estimate the approximation by using a new type of weighted modulus of continuity introduced by *A.D. Gadjiev and A. Aral* (2007) in [9].

1. Introduction

In 1969 Jakimovski and Leviatan [1] introduced a new type of operators P_n by using Appell polynomials as follows. Let $g(u) = \sum_{n=0}^{\infty} a_n u^n$, $g(1) \neq 0$ be an analytic function in the disc $|u| < r$, $(r > 1)$ and $p_k(x) = \sum_{i=0}^{k} a_i \frac{x^{k-i}}{(k-i)!}$, $(k \in \mathbb{N})$ be the Appell polynomials defined by the identity

$$g(u)e^{ux} \equiv \sum_{k=0}^{\infty} p_k(x)u^k. \tag{1.1}$$

We consider the class of functions of exponential type which are defined on the semi-axis and satisfy the property $|f(x)| \leq \beta e^{\alpha x}$ for some finite constants $\alpha, \beta > 0$ and denote the set of functions that satisfy this inequality by $E[0, \infty)$. In [1], the authors considered the operator P_n, with

$$P_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{1.2}$$

for $f \in E[0, \infty)$ and established several approximation properties of these operators, as well as the analogue to Szász's results.

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Remark 1. If $g(1) = 1$ in (1.1) we get $p_k(x) = \frac{x^k}{k!}$, and we recover the well-known classical Favard–Szász operators defined by

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

The following theorem was proved by *B. Wood* in [2].

Theorem 2. *The operator P_n is positive on $[0, \infty)$ if and only if $\frac{a_n}{g(1)} \geq 0$ for $n \in \mathbb{N}$.*

In this paper, we consider the following Chlodowsky [3] type generalization of *Jakimovski–Leviatan operators* given by (1.2):

$$P_n^*(f; x) = \frac{e^{-\frac{n}{b_n}x}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) f\left(\frac{k}{n}b_n\right) \tag{1.3}$$

with b_n a positive increasing sequence with the properties

$$\lim_{n\to\infty} b_n = \infty, \qquad \lim_{n\to\infty} \frac{b_n}{n} = 0 \tag{1.4}$$

and p_k are Appell polynomials defined by (1.1). Recently, some generalizations of (1.2) operators have been considered in [4–6].

The rest of the paper is organized as follows. In Section 2 we obtain some local approximation results for the generalized Jakimovski–Leviatan operators (1.3). In particular, we investigate a Korovkin theorem, the rate of convergence by using the modulus of continuity and a Voronovskaya theorem for these operators. Also some examples in which the approximation of continuous function by using P_n^* operators are given in this section. In Section 3 we study some convergence properties of these operators in weighted spaces with weighted norm on the interval $[0, \infty)$ by using the weighted Korovkin-type theorems, proved by Gadjiev [7,8]. We also prove the estimates of approximation of functions by the operators (1.3) by means of the new type of weighted modulus of continuity, introduced by Gadjiev and Aral [9].

Note that throughout this paper we will assume that the operators P_n^* are positive and we use the following test functions

$$e_i(x) = x^i, \quad i \in \{0, 1, 2, 3, 4\}.$$

2. Local approximation properties of $P_n^*(f; x)$

We denote by $C_E[0, \infty)$ the set of all continuous functions f on $[0, \infty)$ with the property that $|f(x)| \leq \beta e^{\alpha x}$, for all $x \geq 0$ and some positive finite α and β. For a fixed $r \in \mathbb{N}$ we denote by $C_E^r[0, \infty) = \left\{f \in C_E[0, \infty) : f', f'', \ldots, f^{(r)} \in C_E[0, \infty)\right\}$.

Using equality (1.1) and the fundamental properties of the P_n^* operators, one can easily get the following lemmas:

Lemma 3. *From (1.1) we have*

$$\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) = g(1)e^{\frac{n}{b_n}x} = \phi_0(x),$$

$$\sum_{k=0}^{\infty} kp_k\left(\frac{n}{b_n}x\right) = \left(\frac{n}{b_n}g(1)x + g'(1)\right)e^{\frac{n}{b_n}x} = \phi_1(x),$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^2p_k\left(\frac{n}{b_n}x\right) &= \left(g(1)\frac{n^2}{b_n^2}x^2 + \left(g(1) + 2g'(1)\right)\frac{n}{b_n}x + g'(1) + g''(1)\right)e^{\frac{n}{b_n}x} \\ &= \phi_2(x),\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^3p_k\left(\frac{n}{b_n}x\right) &= \left(g(1)\frac{n^3}{b_n^3}x^3 + \left(4g(1) + 3g'(1)\right)\frac{n^2}{b_n^2}x^2\right. \\ &\quad \left. + \left(g(1) + 8g'(1) + 3g''(1)\right)\frac{n}{b_n}x + g'(1) + 4g''(1) + g'''(1)\right)e^{\frac{n}{b_n}x} \\ &= \phi_3(x),\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{\infty} k^4p_k\left(\frac{n}{b_n}x\right) &= \left(g(1)\frac{n^4}{b_n^4}x^4 + \left(10g(1) + 4g'(1)\right)\frac{n^3}{b_n^3}x^3 + \left(14g(1) + 30g'(1) + 6g''(1)\right)\frac{n^2}{b_n^2}x^2\right. \\ &\quad \left. + \left(g(1) + 28g'(1) + 30g''(1) + 4g'''(1)\right)\frac{n}{b_n}x + g'(1) + 14g''(1) + 10g'''(1) + g^{(4)}(1)\right)e^{\frac{n}{b_n}x} \\ &= \phi_4(x).\end{aligned}$$