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Korovkin type approximation theorems proved via $\alpha\beta$-statistical convergence

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ABSTRACT

The concept of statistical convergence was introduced by H. Fast, and studied by various authors. Recently, by using the idea of statistical convergence, M. Balcerzak, K. Dems and A. Komisarski introduced a new type of convergence for sequences of functions called equistatistical convergence. In the present paper we introduce the concepts of $\alpha\beta$-statistical convergence and $\alpha\beta$-statistical convergence of order γ. We show that $\alpha\beta$-statistical convergence is a non-trivial extension of ordinary and statistical convergences. Moreover we show that $\alpha\beta$-statistical convergence includes statistical convergence, λ-statistical convergence, and lacunary statistical convergence. We also introduce the concept of $\alpha\beta$-equistatistical convergence which is a non-trivial extension of equistatistical convergence. Moreover, we prove that $\alpha\beta$-equistatistical convergence lies between $\alpha\beta$-statistical pointwise convergence and $\alpha\beta$-statistical uniform convergence. Finally we prove Korovkin type approximation theorems via $\alpha\beta$-statistical uniform convergence of order γ and $\alpha\beta$-equistatistical convergence of order γ.

1. Introduction

Let K be a subset of $\mathbb{N}$, the set of all natural numbers; then the natural density of the set K is denoted by $\delta(K)$ and defined as

$$\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n}$$

where $K_n := \{k \le n : k \in K\}$. The concept of statistical convergence, which is a generalization of ordinary convergence, has been defined by H. Fast in the following way [1].

A sequence x is said to be statistically convergent to L and denoted by st-$\lim_{n\to\infty} x_n = L$ if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in [1,n] : |x_k - L| \ge \varepsilon\}|}{n} = 0.$$

Recall that a lacunary sequence $\theta = \{k_n\}$ is an increasing integer sequence such that $k_0 = 0$ and $h_n = k_n - k_{n-1} \to \infty$ as $n \to \infty$. A sequence x is said to exhibit lacunary statistical convergence [2] if there exists L such that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in (k_{n-1}, k_n] : |x_k - L| \ge \varepsilon\}|}{h_n} = 0.$$

On the other hand, let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers satisfying

$$\lambda_n \to \infty \quad (n \to \infty), \qquad \lambda_1 = 1, \qquad \lambda_{n+1} \le \lambda_n + 1;$$

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then a sequence x is said to be λ-statistically convergent [3] if there exists L such that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{|\{k \in [n-\lambda_n+1, n] : |x_k - L| \geq \varepsilon\}|}{\lambda_n} = 0.$$

For any non-negative regular matrix $A = (a_{nk})$, statistical convergence was extended to A-statistical convergence by Kolk [4]. A sequence x is said to be A-statistically convergent to L, and denoted by $\mathrm{st}_A\text{-}\lim_{n\to\infty} x_n = L$, if, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{k\in K(\varepsilon)} a_{nk} = 0$$

where $K(\varepsilon) := \{k : |x_k - L| \geq \varepsilon\}$. Statistical convergence, lacunary statistical convergence and λ-statistical convergence are well known examples of A-statistical convergence.

Korovkin type approximation theory was initiated by P. P. Korovkin in [5]. Later, equistatistical convergence was introduced in [6]. Recently, A-statistical convergence and its applications to positive linear operators and Korovkin type approximation theorems have been studied by different authors (see [7–20]).

The main motivation of the present paper is to introduce $\alpha\beta$-statistical convergence methods, which include not only some well known regular matrix methods such as statistical convergence, lacunary statistical convergence and λ-statistical convergence methods but also some non-regular matrix methods. Moreover, we prove Korovkin type approximation theorems via $\alpha\beta$-statistical uniform convergence of order γ and $\alpha\beta$-equistatistical convergence of order γ. Up to now, Korovkin type approximation theorems have been considered only for $\gamma = 1$. But in this paper we give Korovkin type approximation theorems for $0 < \gamma \leq 1$. Therefore all the results obtained here for $0 < \gamma < 1$ are new. On the other hand, the results obtained here for $\gamma = 1$ are non-trivial extensions of some Korovkin type approximation theorems considered by different authors in the past.

2. $\alpha\beta$-statistical convergence

Now let $\alpha(n)$ and $\beta(n)$ be two sequences of positive numbers satisfying the following conditions:

P_1 : α and β are both non-decreasing,

P_2 : $\beta(n) \geq \alpha(n)$,

P_3 : $\beta(n) - \alpha(n) \to \infty$ as $n \to \infty$,

and let Λ denote the set of pairs (α, β) satisfying P_1, P_2 and P_3.

For each pair $(\alpha, \beta) \in \Lambda$, $0 < \gamma \leq 1$ and $K \subset \mathbb{N}$, we define $\delta^{\alpha,\beta}(K, \gamma)$ in the following way:

$$\delta^{\alpha,\beta}(K,\gamma) = \lim_{n\to\infty} \frac{\left|K \cap P_n^{\alpha,\beta}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} \tag{2.1}$$

where $P_n^{\alpha,\beta}$ is the closed interval $[\alpha(n), \beta(n)]$ and $|S|$ represents the cardinality of S. We can state as a consequence of (2.1) the following lemma.

Lemma 1. *Let K and M be two subsets of $\mathbb{N}$ and $0 < \gamma \leq \delta \leq 1$; then, for all $(\alpha, \beta) \in \Lambda$,*

(i) $\delta^{\alpha,\beta}(\phi, \gamma) = 0$,
(ii) $\delta^{\alpha,\beta}(\mathbb{N}, 1) = 1$,
(iii) *if K is a finite set then* $\delta^{\alpha,\beta}(K, \gamma) = 0$,
(iv) $K \subset M \Rightarrow \delta^{\alpha,\beta}(K, \gamma) \leq \delta^{\alpha,\beta}(M, \gamma)$,
(v) $\delta^{\alpha,\beta}(K, \delta) \leq \delta^{\alpha,\beta}(K, \gamma)$.

Now, we are ready to give the following generalization of ordinary convergence.

Definition 1. A sequence x is said to be $\alpha\beta$-statistically convergent of order γ to L, and denoted by $\mathrm{st}^{\gamma}_{\alpha\beta}\text{-}\lim_{n\to\infty} x_n = L$, if, for every $\varepsilon > 0$,

$$\delta^{\alpha,\beta}\left(\{k : |x_k - L| \geq \varepsilon\}, \gamma\right) = \lim_{n\to\infty} \frac{\left|\left\{k \in P_n^{\alpha,\beta} : |x_k - L| \geq \varepsilon\right\}\right|}{(\beta(n) - \alpha(n) + 1)^{\gamma}} = 0.$$

For $\gamma = 1$, we say that x is $\alpha\beta$-statistically convergent to L and this is denoted by $\mathrm{st}_{\alpha\beta}\text{-}\lim_{n\to\infty} x_n = L$.

Remark 1. It is obvious that if $0 < \gamma \leq \delta \leq 1$ and

$$\mathrm{st}^{\gamma}_{\alpha\beta}\text{-}\lim_{n\to\infty} x_n = L$$

then

$$\mathrm{st}^{\delta}_{\alpha\beta}\text{-}\lim_{n\to\infty} x_n = L.$$