Journal of Computational and Applied Mathematics 259 (2014) 194–204

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the approximate controllability of fractional evolution equations with compact analytic semigroup

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ARTICLE INFO

Article history:
Received 31 October 2012
Received in revised form 30 May 2013

Keywords:
Approximate controllability
Fractional differential equations
Compact operators
Semigroup theory

ABSTRACT

In this paper we discuss the approximate controllability of fractional evolution equations involving Caputo fractional derivative. The results are obtained with the help of the theory of fractional calculus, semigroup theory and the Schauder's fixed point theorem under the assumption that the corresponding linear system is approximately controllable. Finally, an example is provided to illustrate the obtained results.

1. Introduction

Controllability is a mathematical problem, which consists of determining the targets to which one can drive the state of some dynamical system by means of a control parameter presenting in the equation. Many physical systems such as quantum systems, fluid mechanic systems, etc. are represented by an infinite number of degrees of freedom and their evolution follows by a partial differential equation. The control theory, namely controllability problems for partial differential equations is a mathematical description of such situations. Controllability of the deterministic and stochastic dynamical control systems in infinite dimensional spaces is well-developed using different kind of approaches in which the details can be found in various papers (see [1–31] and the references therein). Several authors [10,12,20,21] studied the concept of exact controllability for systems represented by nonlinear evolution equations where the fixed point approach is effectively used. Most of the controllability results in infinite dimensional control systems are concerned with semilinear systems consisting of a linear and a nonlinear part. From the mathematical point of view, the problems of exact and approximate controllability are to be distinguished. Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that system can be steered to an arbitrary small neighborhood of final state. Approximately controllable systems are more prevalent and very often approximate controllability is completely adequate in applications [4,7,16]. Therefore, it is important, in fact necessary to study the weaker concept of controllability, namely approximate controllability for nonlinear systems. In the recent literature, there are limited papers on the approximate controllability of the nonlinear evolution systems under different conditions [7,23–25]. Fu and Mei [15] investigated the approximate controllability of semilinear neutral functional differential systems with finite delay. The conditions are established with the help of semigroup theory and fixed point technique under the assumption that the associated linear system is approximately controllable.

In recent years, a considerable interest has been shown in the so-called fractional calculus, which allows us to consider integration and differentiation of any order not necessarily an integer. Fractional calculus has found many applications in physics, chemistry, engineering and control, etc. Derivatives and integrals of fractional order are suitable for the description of properties of various real materials. The most important advantage of using fractional differential equations

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http://dx.doi.org/10.1016/j.cam.2013.06.015

in applications is its nonlocal property which means that the next state of a system depends not only its current state but also all of its historical states. Several researchers studied the existence results of the initial and boundary value problem for fractional differential equations, see ([32,2,18,19] and references therein). The motivation for those works rises from both the development of the theory of fractional calculus itself and the applications of such constructions in various fields of sciences [33,34]. More recently, Zhou and Jiao [35,36], and Yang [37] discussed the existence of mild solutions to fractional evolution and neutral evolution equations in an arbitrary Banach space in which the mild solution is introduced based on the probability density function and semigroup. Furthermore, Wang and Zhou [38] introduced a new mild solution for a class of fractional evolution equations and further the existence of optimal control for the considered problem has been discussed.

Approximate controllability of fractional evolution systems has been considered in [26–28,30,31]. However, it should be emphasized that to the best of our knowledge, the approximate controllability of a semilinear fractional differential and integrodifferential system with compact analytic semigroup in Hilbert spaces has not been investigated yet and it is also the motivation of this paper. In order to fill this gap, in this paper, we study the approximate controllability of a class of control system described by semilinear fractional integrodifferential equations using fixed point techniques, fractional calculus and the assumption that the associated linear system is approximately controllable. Finally, an example is given to illustrate the applications of the theory.

The purpose of this paper is to establish sufficient conditions for the approximate controllability of certain classes of abstract fractional evolution equations with control of the form

$$\begin{cases} {}^{c}D_t^q x(t) = -Ax(t) + Bu(t) + f(t, x(t), Gx(t)), & t \in [0, T], \\ x(0) = x_0, \end{cases} \tag{1}$$

where the state variable x takes values in a Hilbert space X; ${}^{c}D^q$ is the Caputo fractional derivative of order $0 < q < 1$; A is the infinitesimal generator of a C_0 semigroup $S(t)$ of bounded operators on the Hilbert space X; the control function u is given in $\in L_2([0, T], U)$, U is a Hilbert space; B is a bounded linear operator from U into X_α; f is a nonlinear term and will be specified later, and

$$Gx(t) = \int_0^t K(t, s)\, x(s)\, ds$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, [0, \infty))$, $\Delta = \{(t, s) : 0 \le s \le t \le T\}$.

2. Problem formulation and preliminaries

Throughout this paper, unless otherwise specified, the following notations will be used. We assume that X is a Hilbert space with norm $\|\cdot\| := \sqrt{\langle \cdot, \cdot \rangle}$. Let $C([0, T], X)$ be the Banach space of continuous functions from $[0, T]$ into X with the norm $\|x\| = \sup_{t\in[0,T]} \|x(t)\|$, here $x \in C([0, T], X)$. In this paper, we also assume that $-A : D(A) \subset X \to X$ is the infinitesimal generator of a compact analytic semigroup $S(t)$, $t > 0$, of uniformly bounded linear operators in X, that is, there exists $M > 1$ such that $\|S(t)\| \le M$ for all $t \ge 0$. Without loss of generality, let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for any $\alpha > 0$, we can define $A^{-\alpha}$ by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)\, dt.$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of X. Hence we can define $A^\alpha := \left(A^{-\alpha}\right)^{-1}$, which is a closed bijective linear operator in X. It can be shown that each A^α has dense domain and that $D\left(A^\beta\right) \subset D(A^\alpha)$ for $0 \le \alpha \le \beta$. Moreover, $A^{\alpha+\beta}x = A^\alpha A^\beta x = A^\beta A^\alpha x$ for every $\alpha, \beta \in R$ and $x \in D(A^\mu)$ with $\mu := \max(\alpha, \beta, \alpha+\beta)$, where $A^0 = I$, I is the identity in X. (For proofs of these facts we refer to the literature [12,17,34].)

We denote by X_α the Hilbert space of $D(A^\alpha)$ equipped with norm $\|x\|_\alpha := \|A^\alpha x\| = \sqrt{\langle A^\alpha x, A^\alpha x\rangle}$ for $x \in D(A^\alpha)$, which is equivalent to the graph norm of A^α. Then we have $X_\beta \hookrightarrow X_\alpha$, for $0 \le \alpha \le \beta$ (with $X_0 = X$), and the embedding is continuous. Moreover, A^α has the following basic properties.

Lemma 1 *([39]). A^α and $S(t)$ have the following properties*

(i) *$S(t) : X \to X_\alpha$ for each $t > 0$ and $\alpha \ge 0$.*
(ii) *$A^\alpha S(t)x = S(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \ge 0$.*
(iii) *For every $t > 0$, $A^\alpha S(t)$ is bounded in X and there exists $M_\alpha > 0$ such that*

$$\|A^\alpha S(t)\| \le M_\alpha t^{-\alpha}.$$

(iv) *$A^{-\alpha}$ is a bounded linear operator for $0 \le \alpha \le 1$, there exists $C_\alpha > 0$ such that $\left\|A^{-\alpha}\right\| \le C_\alpha$.*

Let us recall the following known definitions in fractional calculus. For more details, see [40,33].