Journal of Computational and Applied Mathematics 259 (2014) 226–232

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The generalized Baskakov type operators

Sevilay Kırcı Serenbay[a,*], Çiğdem Atakut[b], İbrahim Büyükyazıcı[b]

[a] Başkent University, Department of Mathematics Education, 06530 Ankara, Turkey

[b] Ankara University, Faculty of Science, Department of Mathematics, Tandogan 06100, Ankara, Turkey

ARTICLE INFO

Article history:
Received 2 September 2012
Received in revised form 2 July 2013

MSC:
41A25
41A36

Keywords:
Baskakov type operators
Finite sum
Linear positive operator
Approximation

ABSTRACT

The use of Baskakov type operators is difficult for numerical calculation because these operators include infinite series. Do the operators expressed as a finite sum provide the approximation properties? Furthermore, are they appropriate for numerical calculation? In this paper, in connection with these questions, we define a new family of linear positive operators including finite sum by using the Baskakov type operators. We also give some numerical results in order to compare Baskakov type operators with this new defined operator.

1. Introduction and preliminaries

A general construction of Baskakov operators based on a sequence of functions $\{\varphi_n\}$ $(n = 1, 2, \ldots)$ $\varphi_n : C \to C$, having the following properties

(i) For every $n = 1, 2, \ldots$ φ_n is analytic on a domain D_n, containing the disc $B_n = \{z \in C : |z - b_n| \le b_n\}$, $\lim_{n\to\infty} b_n = \infty$;

(ii) $\varphi_n(0) = 1$ $(n = 1, 2, \ldots)$;

(iii) φ_n $(n = 1, 2, \ldots)$ is completely monotone on $[0, b_n]$, i.e., $(-1)^k \varphi_n^{(k)}(x) \ge 0$ for any $k = 0, 1, 2, \ldots$;

(iv) there exists a positive integer $m(n)$, such that

$$\varphi_n^{(k)}(x) = -n\varphi_{m(n)}^{(k-1)}(x)\left(1 + \alpha_{k,n}(x)\right), \quad x \in [0, b_n] \quad (n, k = 1, 2, \ldots)$$

where $\alpha_{k,n}(0)$ converges to zero for $n \to \infty$ uniformly in k;

(v) $\lim_{n\to\infty} \frac{n}{m(n)} = 1$.

Under these conditions we will consider the following Baskakov type operators

$$L_n(f, x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!}\varphi_n^{(k)}(x) f\left(\frac{k}{n}\right), \quad 0 \le x \le b_n. \tag{1}$$

It is obvious that L_n translated a continuous function with the growth condition $f(x) =: O\left(x^2\right)$ at infinity to such a type of function, which may be seen from the properties (2)–(4).

* Corresponding author. Tel.: +90 312 482 5863.
E-mail addresses: sevilaykirci@gmail.com, kirci@baskent.edu.tr (S.K. Serenbay), atakut@science.ankara.edu.tr (Ç. Atakut), ibuyukyazici@gmail.com (İ Büyükyazıcı).

0377-0427/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.cam.2013.08.004

Lemma 1. *The following equalities hold:*

$$L_n(1,x) = 1, \tag{2}$$

$$L_n(t,x) = \left(1+\alpha_{1,n}(0)\right)x, \tag{3}$$

$$L_n\left(t^2,x\right) = \frac{m(n)}{n}\left(1+\alpha_{1,m(n)}(0)\right)(1+\alpha_{2,n}(0))x^2 + \frac{(1+\alpha_{2,n}(0))}{n}x. \tag{4}$$

Proof. Since φ_n $(n = 1, 2, \ldots)$ be analytic on a domain D_n, we have

$$\varphi_n(y) = \sum_{k=0}^{\infty}\frac{(y-x)^k}{k!}\varphi_n^{(k)}(x).$$

By condition (ii), for $y = 0$ we get

$$L_n(1,x) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k)}(x) = 1.$$

Now, we consider the case $L_n(t,x)$ as follows:

$$\begin{aligned} L_n(t,x) &= \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k)}(x)\frac{k}{n} \\ &= \frac{-x}{n}\sum_{k=1}^{\infty}\frac{(-x)^{k-1}}{(k-1)!}\varphi_n^{(k)}(x) \\ &= \frac{-x}{n}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x). \end{aligned}$$

From the equality $\varphi_n^{(r)}(0) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+r)}(x)$ we have

$$\varphi_n'(0) = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k+1)}(x)$$

therefore we get,

$$L_n(t,x) = \frac{-x}{n}\varphi_n'(0).$$

From condition (iv), we obtained the desired result. (4) can be proved similarly. ■

When $b_n = b$ in (1), in the case when all functions φ_n, $n = 1, 2, \ldots$, are analytic on the fixed disc $B = \{z \in C : |z-b| \le b\} \subset D$ where D is a domain, the sequence of operators (1) were investigated by many authors (see, for example [1–5]).

But all of these investigations are devoted to the problem of approximation of a function belonging to the class mentioned above and we do not know of any further result on approximation theorems in polynomial weighted spaces, given in [6] for a special Baskakov operator, which may be obtained from (1) in the case of

$$\varphi_n(x) = \frac{1}{(1+x)^n}, \quad x \ge 0,\ n = 1, 2, \ldots.$$

In [7], Gadziev and Atakut investigated the approximation of continuous functions having polynomial growth at infinity, by the operator given in (1). They also gave an estimate for a difference $|L_n(f,x) - f(x)|$ on any finite interval through the modulus of continuity of a function f and the theorem on weighted approximation on all positive semi-axes.

Note that a weighted Korovkin's type theorem was proven in [8,9] and we need a special case of this theorem.

Let $B_{2m}[0,\infty)$ be the space of all functions, satisfying the inequality

$$|f(x)| \le M_f\left(1+x^{2m}\right), \quad x \ge 0,\ m \in \mathbb{N} \tag{5}$$

where M_f is constant, depending on a function f and let $C_{2m}[0,\infty)$ consist of all continuous functions belonging to $B_{2m}[0,\infty)$. Let also $C_{2m}^0[0,\infty)$ be a subset of functions in $C_{2m}[0,\infty)$ for which

$$\lim_{x\to\infty}\frac{f(x)}{1+x^{2m}} \tag{6}$$

exists finitely.