Journal of Computational and Applied Mathematics 259 (2014) 312–326

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On boundary conditions for the gravity wave equations

Andrei Bourchtein *, Ludmila Bourchtein

Institute of Physics and Mathematics, Pelotas State University, Brazil

ARTICLE INFO

Article history:
Received 20 August 2012

MSC:
65M30
76B03
86A10

Keywords:
Gravity wave equations
Shallow water equations
Initial–boundary value problems
Well posed problems
Energy estimates

ABSTRACT

In this study, the reasons why mathematically well posed problems for gravity wave equations with quite natural initial and boundary conditions can produce physically meaningless solutions are examined. The mechanism of generating such solutions is analyzed and general conditions on initial and boundary functions are found under which the solutions have at least linear growth with respect to time. Different examples of smooth bounded input functions are given, which lead to unbounded growth of the respective solutions. The same problem can rise in numerical models for one- and two-dimensional gravity wave and shallow water equations, but origin of the problem is hard to be found without analysis of the primitive differential problems. Based on the performed analysis and numerical experiments, some recommendations for choosing the boundary conditions are given to avoid this unphysical behavior of differential and numerical solutions.

1. Introduction and problem formulation

Let us consider the one-dimensional linear system of the gravity waves on the bounded domain [1,2]

$$u_t = -\Phi_x, \qquad \Phi_t = -c^2 u_x, \quad x \in [a, b], \; t \in [0, T]; \tag{1.1}$$

$$u(0, x) = u_0(x), \qquad \Phi(0, x) = \Phi_0(x), \quad x \in [a, b]; \tag{1.2}$$

$$a_1 u(t, a) + a_2 \Phi(t, a) = g_a(t), \qquad b_1 u(t, b) + b_2 \Phi(t, b) = g_b(t), \quad t \in [0, T], \tag{1.3}$$

where t and x are the time and space variables, respectively, u and Φ are unknown functions, $c > 0$ is the propagation speed of the gravity waves, a_1, a_2 and b_1, b_2 are the constants such that $a_1 \neq -c\, a_2$ and $b_1 \neq c\, b_2$, that is, the linear combinations in the left-hand sides of the boundary conditions do not represent the outgoing characteristic variables, the subscripts t and x denote derivatives and other subscripts are the indices.

To ensure the smoothness of solutions of class $C^1(x, t)$ we impose the following compatibility conditions:

$$a_1 u_0(a) + a_2 \Phi_0(a) = g_a(0), \qquad b_1 u_0(b) + b_2 \Phi_0(b) = g_b(0), \tag{1.4}$$

$$a_1 \Phi_{0_x}(a) + a_2 c^2 u_{0_x}(a) = -g_{a_t}(0), \qquad b_1 \Phi_{0_x}(b) + b_2 c^2 u_{0_x}(b) = -g_{b_t}(0). \tag{1.5}$$

Further in the text we will need also the $C^2(x, t)$ solutions, which implies two more compatibility conditions:

$$a_1 c^2 u_{0_{xx}}(a) + a_2 c^2 \Phi_{0_{xx}}(a) = g_{a_{tt}}(0), \qquad b_1 c^2 u_{0_{xx}}(b) + b_2 c^2 \Phi_{0_{xx}}(b) = g_{b_{tt}}(0). \tag{1.6}$$

Eqs. (1.1) can be considered as an isolated system for one-dimensional surface gravity waves or as a part of a more complex hydrodynamic system, for example, Euler equations of atmosphere–ocean dynamics, with u and Φ being the

* Corresponding author. Tel.: +55 53 32285445; fax: +55 53 32757343.
E-mail addresses: burstein@terra.com.br, bourchtein@gmail.com (A. Bourchtein).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2012.08.033

velocity of fluid particles and geopotential height, that is, the elevation of fluid surface multiplied by the gravitational acceleration [1,2].

Due to their physical meaning, the solutions of (1.1) are expected to be bounded on $\Omega_\infty = [a,b] \times [0,\infty]$, that is,

$$\max_{\Omega_\infty} \left(|u|, |\Phi| \right) \le C = \text{const} \tag{1.7}$$

if the natural initial and boundary conditions are provided (at ∞ the values are considered in the limit sense). Under natural initial and boundary functions we mean those which correspond to the physical nature of fluid dynamics problems, that is, smooth bounded on $[a,b]$ and $[0,\infty]$, respectively, functions. Evidently, condition (1.7) is equivalent to uniform boundness of solutions to (1.1)–(1.3) with respect to parameter T.

Since the system (1.1) is strictly hyperbolic [3,4], the initial-boundary values problem (1.1)–(1.3) is well posed, that is, it has the unique smooth solution satisfying energy estimate [3,4]. For the boundary conditions in general form (1.3) the energy estimate can be deduced by the method pointed out in [3,4]. First, by introducing the characteristic functions

$$p = c\,u + \Phi, \qquad q = c\,u - \Phi \tag{1.8}$$

the primitive system (1.1)–(1.3) is transformed to

$$p_t = -cp_x, \qquad q_t = cq_x, \quad x \in [a,b],\ t \in [0,T]; \tag{1.9}$$

$$p(0,x) = p_0, \qquad q(0,x) = q_0, \quad x \in [a,b]; \tag{1.10}$$

$$p(t,a) = -s_a q(t,a) + \hat{g}_a, \qquad q(t,b) = -s_b p(t,b) + \hat{g}_b, \quad t \in [0,T] \tag{1.11}$$

with

$$p_0 = c\,u_0 + \Phi_0, \qquad q_0 = c\,u_0 - \Phi_0,$$
$$s_a = \frac{a_1 - a_2 c}{a_1 + a_2 c}, \qquad s_b = \frac{b_1 + b_2 c}{b_1 - b_2 c}, \qquad \hat{g}_a = \frac{2c}{a_1 + a_2 c} g_a, \qquad \hat{g}_b = \frac{2c}{b_1 - b_2 c} g_b.$$

Then boundary coefficients should be normalized in an appropriate way by using auxiliary function

$$d(x) = d(a)\frac{x-b}{a-b} + d(b)\frac{x-a}{b-a}, \qquad |d(a)s_a| \le \frac{1}{2}, \qquad \left|\frac{s_b}{d(b)}\right| \le \frac{1}{2}, \quad d(a) > 0,\ d(b) > 0 \tag{1.12}$$

such that the functions $\tilde{p} = dp$, $\tilde{q} = q$ satisfy the following system:

$$\tilde{p}_t = -c\tilde{p}_x + c_a\tilde{p}, \qquad \tilde{q}_t = c\tilde{q}_x, \quad x \in [a,b],\ t \in [0,T]; \tag{1.13}$$

$$\tilde{p}(0,x) = \tilde{p}_0, \qquad \tilde{q}(0,x) = \tilde{q}_0, \quad x \in [a,b]; \tag{1.14}$$

$$\tilde{p}(t,a) = -\tilde{s}_a\tilde{q}(t,a) + \tilde{g}_a, \qquad \tilde{q}(t,b) = -\tilde{s}_b\tilde{p}(t,b) + \tilde{g}_b, \quad t \in [0,T], \tag{1.15}$$

where

$$c_a = cd_x/d, \qquad \tilde{p}_0 = dp_0, \qquad \tilde{q}_0 = q_0, \qquad \tilde{s}_a = d(a)s_a,$$
$$\tilde{s}_b = s_b/d(b), \qquad \tilde{g}_a = d(a)\hat{g}_a, \qquad \tilde{g}_b = \hat{g}_b.$$

Inequalities in (1.12) imply

$$\tilde{p}^2(a) - \tilde{q}^2(a) \le 2\tilde{g}_a^2 - \left(1 - 2\tilde{s}_a^2\right)\tilde{q}^2(a) \le 2\tilde{g}_a^2,$$

and

$$\tilde{q}^2(b) - \tilde{p}^2(b) \le 2\tilde{g}_b^2 - \left(1 - 2\tilde{s}_b^2\right)\tilde{p}^2(b) \le 2\tilde{g}_b^2.$$

Therefore, for the last system we have

$$\left(\int_a^b \tilde{p}^2 + \tilde{q}^2 dx\right)_t = c\left(\tilde{q}^2 - \tilde{p}^2\right)\Big|_a^b + 2\int_a^b c_a\tilde{p}^2 dx \le 2c\tilde{g}_a^2 + 2c\tilde{g}_b^2 + 2\int_a^b c_a\tilde{p}^2 dx$$

or

$$\left(\int_a^b \tilde{p}^2 + \tilde{q}^2 dx\right)_t \le 2c\left(\tilde{g}_a^2 + \tilde{g}_b^2\right) + \tilde{c}\int_a^b \tilde{p}^2 + \tilde{q}^2 dx, \tag{1.16}$$

where

$$\tilde{c} = 2c\frac{|d(b) - d(a)|}{b-a}\frac{1}{\min\left(d(a), d(b)\right)}.$$

Integrating (1.16) with respect to t, we obtain

$$\int_a^b \tilde{p}^2 + \tilde{q}^2 dx \le 2ce^{\tilde{c}t}\int_0^t \left(\tilde{g}_a^2 + \tilde{g}_b^2\right)d\tau + e^{\tilde{c}t}\int_a^b \tilde{p}_0^2 + \tilde{q}_0^2 dx, \tag{1.17}$$