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Finite difference methods for the Infinity Laplace and p-Laplace equations

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ABSTRACT

We build convergent discretizations and semi-implicit solvers for the Infinity Laplacian and the game theoretical p-Laplacian. The discretizations simplify and generalize earlier ones. We prove convergence of the solution of the Wide Stencil finite difference schemes to the unique viscosity solution of the underlying equation. We build a semi-implicit solver, which solves the Laplace equation as each step. It is fast in the sense that the number of iterations is independent of the problem size. This is an improvement over previous explicit solvers, which are slow due to the CFL condition.

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1. Introduction

The Infinity Laplacian equation is at the interface of the fields of analysis, nonlinear elliptic Partial Differential Equations (PDEs), and probabilistic games. It was first studied in the late 1960s by the Swedish mathematician Gunnar Aronsson [1–3], motivated by the classical analysis problem of building Lipschitz extensions of a given function. Aronsson found non-classical solutions, but a rigorous theory of weak solutions was not yet available. It took a few decades until analytical tools were developed to study the equation rigorously, and computational tools were developed which made numerical solution of the equation possible.

In the last decade, PDE theorists established existence and uniqueness, and regularity results. The theory of viscosity solutions [4] is the appropriate one for studying weak solutions to the PDE. But the general uniqueness theory did not apply to this very degenerate equation, so proving uniqueness required a new approach. The first uniqueness result was due to Jensen [5], followed by a different proof by Barles and Busca [6]. Later, the connection with finite difference equations was exploited by Armstrong and Smart, and they were to give a short uniqueness proof for the PDE [7]. The differentiability of solutions remained an open question for some time. The first result was obtained by [8] in two dimensions, followed by [9–11] in general dimensions.

The first convergent difference scheme was presented in [12]. A numerical scheme using the extension property can be found in [13]. Two different numerical methods were derived in [11], one adapted the monotone scheme in [12] to the standard Infinity Laplacian (which is homogeneous degree two in ∇u), the second, quite different, used the variational structure of a regularized PDE.

Earlier work by LeGruyer [14] proved uniqueness for a related finite difference equation. The proof is a generalization of the uniqueness proof for linear elliptic finite difference schemes from [15].

A group of probabilists, Peres–Schramm–Sheffield–Wilson [16], studying a randomized version of a marble game called Hex found a connection with the Infinity Laplacian equation. This connection gives an interpretation of the equation as a two player random game. This is related to work by Kohn–Serfaty [17] who found an interpretation of the equation for

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motion of level sets by mean curvature [18,19] as a deterministic two player game. This equation was also interpreted as a nonlinear average [20] for the purpose of finite difference schemes. Deterministic game interpretations for more general PDEs followed in [21]. The connection between these various game interpretations was further studied in [22].

The rich connection between games, finite difference schemes, and nonlinear elliptic PDEs is now much better understood. There have been a number of works in this area, in particular on the game theoretical p-Laplacian. The probabilistic games interpretation can be found in [23] (see also [24]). Related works include biased games which correspond to a gradient term [25].

This article will further exploit the connection between games, finite difference schemes, and nonlinear elliptic PDE, by building convergent finite difference schemes which are consistent with the game interpretation. The existence and uniqueness results are now established, but efficient numerical solution of the equation remains a challenge. The original convergent scheme proposed in [12] converged, but is not efficient: as the grid size grows, so does the number of iterations required to find the solution. This article improves and simplifies the original discretization, and also finds fast solution methods. It also generalizes the scheme and the solvers to the game-theoretical p-Laplacian, which is a convex combination of the Laplacian and the Infinity Laplacian.

1.1. Introduction to numerical methods for degenerate elliptic PDEs

There are two major challenges in building numerical solvers for nonlinear and degenerate elliptic Partial Differential Equations (PDEs). The first challenge is to build convergent approximations, usually by finite difference schemes. The second challenge is to build efficient solvers.

The approximation theory developed by Barles and Souganidis [26] provides criteria for the convergence of approximation schemes: monotone, consistent, and stable schemes converge to the unique viscosity solution of a degenerate elliptic equation. But this work does not indicate how to build such schemes, or how to produce fast solvers for the schemes. It is not obvious how to ensure that schemes satisfy the comparison principle. The class of schemes for which this property holds was identified in [27], and was called *elliptic*, by analogy with the structure condition for the PDE.

An important distinction for this class of equations is between first order (Hamilton–Jacobi) equations, and the second order (nonlinear elliptic) case. The theory of viscosity solutions [4] covers both cases, but the numerical methods are quite different. In the first order case, where the method of characteristics is available, there are some exact solutions formulas (e.g. Hopf–Lax) and there is a connection with one dimensional conservation laws [28]. The second order case has more in common with divergence-structure elliptic equations, but because of the degeneracy or nonlinearity, many of the tools from the divergence-structure case (e.g. finite elements, multi-grid solvers) have not been successfully applied.

In the first order case, there is much more work on discretizations and fast solvers. For Hamilton–Jacobi equations, which are first order nonlinear PDEs, monotonicity is necessary for convergence. Early numerical papers studied explicit schemes for time-dependent equations on uniform grids [29,30]. These schemes have been extended to higher accuracy schemes, which include second order convergent methods, the central schemes [31], as well as higher order interpolation methods, the ENO schemes [32]. Semi-Lagrangian schemes take advantage of the method of characteristics to prove convergence [33]. These have been extended to the case of differential games [34]. Two classes of fast solvers have been developed, fast marching [35], and fast sweeping [36], The fast marching and fast sweeping methods give a fast solution method for first order equations: both take advantage of the method of characteristics, which is not available in the second order case.

There is much less work in the second order degenerate elliptic equations. The equation for motion by mean curvature [18,19] has been extensively studied. There is an enormous literature on this equation, but we just consider closely related references. The connection with games was already mentioned above. Numerical schemes include [37,38]. In the case of motion by mean curvature, the equation is time-dependent, so a fast solver would allow larger time steps. For this equation, a semi-implicit solver has been built by Smereka [39]. The idea from the Smereka paper will be adapted in this work to build fast solvers for the Infinity Laplacian. Another equation in this class is the Hamilton–Jacobi–Bellman equations, for the value function of a stochastic control problem. Applications include portfolio optimization and option pricing in mathematical finance. Numerical works include the early paper [40,41], and a paper on fast solvers [42].

For uniformly elliptic PDEs, monotone schemes are not *necessary* for convergence (for example most higher order Finite Element Methods are not monotone). But for fully nonlinear or degenerate elliptic, the only convergence proof currently available requires monotone schemes. One way to ensure monotone schemes is to use Wide Stencil Finite difference schemes, this has been done for the equation for motion by mean curvature, [20], for the Infinity Laplace equation [12], for functions of the eigenvalues [43], for Hamilton–Jacobi–Bellman equations [44], and for the convex envelope [45]. Even for linear elliptic equations, a Wide Stencil scheme may be necessary to build a monotone scheme [15]. In some cases, simple finite difference schemes, with minor modifications, can give good results, as is the case for the Monge–Ampère equation [46]. But we show below that simple finite difference schemes are not convergent for the Infinity Laplace equations.

The second challenge, which is quite distinct from the first, is to build *solvers* for the finite difference schemes. For fixed values of h, the finite difference scheme is a finite dimensional nonlinear algebraic equation which must be solved. Building solvers demands very different techniques, and little progress has been made, in part, due to the fact that the discrete equations can be non-differentiable, which precludes the use of Newton's method. To date, the only general solver available is a fixed point iteration, which corresponds to solving the parabolic version of the equation for long time. This method is restricted by a nonlinear version of the CFL condition [27], which means the number of iterations required to solve the

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