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hp-FEM convergence for unilateral contact problems with Tresca friction in plane linear elastostatics

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Dedicated to Klaus Böhmer on the occasion of his seventy fifth birthday

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ABSTRACT

This paper is concerned with the convergence of the *hp*-version of the finite element method (*hp*-FEM) for some nonsmooth unilateral problems in linear elastostatics. We consider in particular the deformation of an elastic body unilaterally supported by a rigid foundation, admitting Tresca friction (given friction) along the rigid foundation, solely subjected to body forces and surface tractions without being fixed along some part of its boundary. For the discretization of the unilateral constraint and the nonsmooth friction functional we employ Gauss–Lobatto quadrature. We show convergence of the *hp*-FEM approximations for mechanically definite problems without imposing any regularity assumption. Moreover we treat the coercive case, when the body is fixed along some part of the boundary. Based on an abstract Céa–Falk estimate and operator interpolation arguments, we establish an a priori error estimate in the energy norm under a reasonable regularity assumption.

1. Introduction

In this paper we investigate the *hp*-version of the finite element method (*hp*-FEM) applied to unilateral contact with Tresca friction in planar linear elastostatics. As already proposed by Panagiotopoulos [1] a fixed point approach to unilateral contact problems obeying the more realistic Coulomb law leads to a sequence of Tresca frictional unilateral contact problems. This approach has been recently substantiated by Dostál, Haslinger and Kučera in [2,3], who implemented the fixed point method by novel splitting techniques. Thus the efficient numerical solution of Tresca unilateral contact problems remains an interesting topic of research.

While the analysis of the standard *h*-version FEM for nonsmooth unilateral contact problems is well documented in the literature (see [4–10] and the references listed therein), the situation is less favorable for the *p*-version and particularly the *hp*-version of the FEM, where the approximation properties of spaces of piecewise polynomials are quantified in terms of both the local mesh size and the local polynomial degree. For the closely related variational inequalities of the first kind arising from non-frictional Signorini and unilateral contact problems, Maischak and Stephan analyzed *hp*-boundary element methods (*hp*-BEM) in [11,12] and obtained convergence rates under certain regularity assumptions on the exact solution. Then Chernov, Maischak and Stephan [13] provided results for the frictional two-body contact problem in the *hp*-BEM; however, the variational crimes associated with approximating the nondifferentiable friction functional j, which is clearly necessary in a high order context, were not addressed. On the other hand, Dörsek and Melenk, based on an analysis in Besov spaces, derived in [14] for the *hp*-FEM approximation of a pure frictional contact problem (without unilateral boundary conditions) an a priori error estimate consisting of a polynomial error term and an additional log error term. An a priori error estimate without such an additional log error term was earlier given by the author in [15], albeit for the *p*-BEM approximation of a simplified scalar model problem extracted from the full friction unilateral contact problem.

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Let us also refer to [16] for a recent numerical study and comparison results for the *h*-, *p*-, *hp*-version of the FEM for the solution of the unilateral frictionless 2*D* Hertzian contact problem. Moreover, Dörsek and Melenk were able to demonstrate numerically in some friction model problems in [17,14] that the exponential convergence of the *hp*-FEM, well-known for linear boundary value problems, can also obtained for this class of nonlinear free boundary value problems, if an appropriate adaptive strategy is used.

In the present paper, we focus on two issues. Firstly, we apply the discretization theory of [6] and its extension to semicoercive variational inequalities of the second type in [18]. We combine this theory with Gauss–Lobatto quadrature for the discretization of the unilateral constraint and the nonsmooth friction functional. Thus we can show convergence of the *hp*-FEM approximations on quadrilaterals for semicoercive unilateral Tresca friction problems without imposing any regularity assumption. Secondly we treat the coercive case, when the body is fixed along some part of its boundary. Here based on an abstract Céa–Falk estimate taken from [15] and operator interpolation arguments, we establish an *hp*-FEM a priori error estimate on quadrilaterals in the energy norm under a reasonable regularity assumption. This error estimate sharpens the error bound given earlier by [14] for the pure frictional contact problem and gives the same *p*-FEM error order on quadrilaterals as in [11] for the frictionless unilateral contact problem. Moreover we close here a gap in the proof of the consistency error in [15] which was communicated to the author by J.M. Melenk. Thus we complement and extend the convergence analysis given in [13–15,12] in several respects.

The plan of the paper is as follows. The next Section 2 presents the variational problem, its *hp*-FEM approximation, and collects preliminary material. The main results are in Sections 3 and 4; in Section 3 we establish *hp*-norm convergence without a regularity assumption, in Section 4 we prove an a priori error bound under a reasonable regularity assumption. Finally in Section 5 we give some concluding remarks and an outlook.

2. The unilateral frictional contact problem and its *hp*-FEM approximation

Let us consider an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary Γ that splits into three disjoint parts $\Gamma_0, \Gamma_T, \Gamma_c$ such that $\Gamma = \overline{\Gamma_0} \cup \overline{\Gamma_T} \cup \overline{\Gamma_c}$. Zero displacements are prescribed on Γ_0, surface tractions $\underline{T} \in (L^2(\Gamma_T))^2$ act on Γ_T, and on the part Γ_c unilateral contact and Tresca friction conditions between the body and a perfectly rigid foundation hold. Thus Γ_c contains the free boundary of the unilateral contact. In the model of Tresca friction (given friction) one assumes a known slip bound $g \in L^\infty(\Gamma_c),\ g \geq 0$. Moreover, the body is subject to body forces $\underline{F} \in (L^2(\Omega))^2$. To make the contact problem meaningful, we assume meas $(\Gamma_c) > 0$, but we do not require meas $(\Gamma_0) > 0$.

We denote by $H^s()$ the usual Sobolev spaces on Ω or on parts of Γ with norms defined using the Slobodeckij seminorms. We also use the short $\underline{H}^s = (H^s)^2$ for the vectorial Sobolev spaces. In particular, we have the space of *virtual displacements*

$$\mathcal{V} = \{\underline{v} \in \underline{H}^1(\Omega) \mid \gamma_0\, \underline{v} = 0\},$$

where $\gamma_o = \gamma_{\Gamma 0} : \underline{H}^1(\Omega) \to \underline{H}^{\frac{1}{2}}(\Gamma_0)$ is the trace map onto Γ_0, and its convex closed subset of *kinematically admissible* displacements

$$\mathcal{K} = \{\underline{v} \in \mathcal{V} \mid (\gamma_c\, \underline{v})_n \leq d\}.$$

Here, likewise $\gamma_c = \gamma_{\Gamma c} : \underline{H}^1(\Omega) \to \underline{H}^{\frac{1}{2}}(\Gamma_c) \subset (L^2(\Gamma_c))^2$, further $d \in C(\overline{\Gamma_c})$, $d \geq 0$ is the initial gap between the body and the rigid foundation, and with the unit outer normal $\underline{n} \in (L^\infty(\Gamma))^2$ to the boundary, a vector field $\underline{w}$ at the boundary has its normal component $w_n = \underline{w} \cdot \underline{n}$ and its normal component $\underline{w}_t = \underline{w} - w_n\, \underline{n}$.

Adopting standard notations from linear elasticity, $\varepsilon(\underline{v}) = \frac{1}{2}(\nabla \underline{v} + \nabla \underline{v}^T)$ denotes the small strain tensor to the displacement field $\underline{v}$ and $\sigma(\underline{v}) = C : \varepsilon(\underline{v})$ the stress tensor. Here, C is the Hooke tensor, assumed to be uniformly positive definite with L^∞ coefficients. This leads to the bilinear form, linear functional, sublinear functional, and to the *total potential energy* of the body, respectively,

$$a(\underline{u}, \underline{v}) = \int_\Omega \varepsilon(\underline{u}) : C : \varepsilon(\underline{v})\, dx,$$

$$l(\underline{v}) = \int_\Omega \underline{F} \cdot \underline{v}\, dx + \int_{\Gamma_T} \underline{T} \cdot \underline{v}\, ds,$$

$$j(\underline{v}) = \int_{\Gamma_c} g\, |\underline{v}_t|\, ds,$$

$$J(\underline{v}) = \frac{1}{2} a(\underline{v}, \underline{v}) - l(\underline{v}) + j(\underline{v}).$$

In these terms, the *variational formulation* of the unilateral contact problem with Tresca friction reads as follows:
Find a minimizer $\underline{u} \in \mathcal{K}$ of the functional $J(\underline{v}),\ \underline{v} \in \mathcal{K}$!
Another equivalent formulation is the *variational inequality* problem (π) *of second kind*: Find $\underline{u} \in \mathcal{K}$ such that for all $\underline{v} \in \mathcal{K}$,

$$a(\underline{u}, \underline{v} - \underline{u}) + j(\underline{v}) - j(\underline{u}) \geq l(\underline{v} - \underline{u}). \tag{1}$$