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An efficient algorithm for solving Fredholm integro-differential equations with weakly singular kernels

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HIGHLIGHTS

- The PHPM is used to solve equations with weakly singular kernels.
- The concrete algorithm and the selection of parameters are described clearly.
- The Gauss rule and low-order interpolation are used to improve the calculation speed.
- A high accuracy can be obtained by our method if the exact solution is smooth enough.

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ABSTRACT

In this paper, an effective method based upon the piecewise homotopy perturbation method (PHPM) is proposed for finding the numerical solutions of Fredholm integro-differential equations with weakly singular kernels. For these equations, the traditional homotopy perturbation method is divergent. So we present a modified homotopy perturbation method for solving this kind of equation. The advantages of the approach of this paper lie in the feature that this technique not only achieves an approximate solution with high accuracy, but also improves the calculation speed. Two examples are given to show the effectiveness and convenience of this method.

1. Introduction

Let $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{N} = \{1, 2, \ldots\}$. In this paper, we will present an efficient method for obtaining the numerical solution of boundary value problems with weakly singular kernels given by

$$u^{(n)}(t) = \sum_{i=0}^{n_0} a_i(t)u^{(i)}(t) + \sum_{i=0}^{n_0} \int_0^b \frac{K_i(t,s)}{|t-s|^{\alpha_i}} u^{(i)}(s)\mathrm{d}s + f(t), \quad 0 \le t \le b, \tag{1}$$

$$\sum_{i=0}^{n-1} [\alpha_{ij} u^{(i)}(0) + \beta_{ij} u^{(i)}(b)] = 0, \quad j = 1, 2, \ldots, n, \tag{2}$$

where u is the function to be determined, $n \in \mathbb{N}$, $0 \le n_0 \le n-1$, $0 \le \alpha_i < 1$ for every $i = 0, 1, 2, \ldots, n_0$ and $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$ for every $i = 0, 1, \ldots, n-1$ and $j = 1, 2, \ldots, n$. We assume that $a_i, f \in C[0, b]$, $K_i \in C([0, b] \times [0, b])$. Furthermore, the problem given by (1) and (2) is proved to be uniquely solvable in [1].

Weakly singular integral equations are used in modeling various physical processes. These equations arise in heat conduction problems [2], elasticity and fracture mechanics [3], potential problems, the Dirichlet problems, radiative equilibrium [4] and so on. It is difficult to solve these equations analytically. Hence, numerical solutions are required.

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In recent years, equations of this form have been approached by different methods including the spline collocation method [1,5], the discrete collocation method [6–8], the discrete Galerkin method [9,10], the Legendre multiwavelets method [11] and the piecewise polynomial collocation method with graded meshes [12] to determine the approximate solutions. However, when these methods were expected to improve the accuracy of the numerical solution, the calculation became more complicated.

In this paper, we will use an efficient method named the piecewise homotopy perturbation method (PHPM) to solve Eq. (1) with (2). The homotopy perturbation method (HPM) [13,14] was first proposed by He in 1999 as a modification of traditional perturbation methods. In most cases, a rapidly convergent series solution can be obtained by this method. To our certain knowledge, the HPM has been applied successfully to linear and nonlinear oscillators [15,16], nonlinear boundary value problems [17], partial differential equation [18–20], fractional partial differential equations [21], delay differential equation [22] and other fields. That is because this method deforms a difficult problem into a simple solving problem. However, the traditional HPM may be divergent when solving Eq. (1) with (2). So the aim of this work is to develop an efficient algorithm called the piecewise HPM (PHPM) through modifying the HPM for finding the approximate solution of Eq. (1) with (2).

The paper is organized as follows. In Section 2, the homotopy perturbation method is introduced. Section 3 is the main content, including the principle of the PHPM, the convergence proof and error estimation. Section 4 describes the details of the proposed algorithm. Also two examples are given to show the effectiveness and convenience of the PHPM. A brief conclusion completes this paper in Section 5.

2. Analysis of the homotopy perturbation method

This section is devoted to reviewing the HPM by considering the Fredholm integro-differential equation (1) with (2).

To explain the HPM, we consider Eq. (1) as

$$L(u) = u^{(n)}(t) - \sum_{i=0}^{n_0} a_i(t)u^{(i)}(t) - \sum_{i=0}^{n_0} \int_0^b \frac{K_i(t,s)}{|t-s|^{\alpha_i}} u^{(i)}(s)\mathrm{d}s - f(t) = 0, \tag{3}$$

where L is a linear operator. Define a convex homotopy $H(v,p)$ by

$$H(v,p) = (1-p)F(v) + pL(v) = 0, \quad p \in [0,1], \tag{4}$$

where $F(v) = v^{(n)}(t) - f(t)$. It is obvious that

$$H(v,0) = F(v) = 0, \qquad H(v,1) = L(v) = 0, \tag{5}$$

and the process of changing p from 0 to 1 is just that of changing v from v_0 to u.

According to the HPM, we can use the embedding parameter p as a "small parameter", and applying the perturbation technique [23], we can assume that the solution of (4) can be written as a power series in p:

$$v = u_0 + pu_1 + p^2u_2 + \cdots, \tag{6}$$

and when $p \to 1$, the exact solution of (3) is obtained as

$$u = \lim_{p\to 1} v = u_0 + u_1 + u_2 + \cdots. \tag{7}$$

Substituting (6) into (4) and equating the terms with identical powers of p, we have

$$p^0 : u_0^{(n)}(t) = f(t), \qquad \sum_{i=0}^{n-1}[\alpha_{ij}u_0^{(i)}(0) + \beta_{ij}u_0^{(i)}(b)] = 0, \quad j = 1,2,\ldots,n, \tag{8}$$

$$p^j : u_j^{(n)}(t) = \sum_{i=0}^{n_0} a_i(t)u_{j-1}^{(i)}(t) + \sum_{i=0}^{n_0}\int_0^b \frac{K_i(t,s)}{|t-s|^{\alpha_i}} u_{j-1}^{(i)}(s)\mathrm{d}s, \tag{9}$$

$$\sum_{i=0}^{n-1}[\alpha_{ij}u_j^{(i)}(0) + \beta_{ij}u_j^{(i)}(b)] = 0, \quad j = 1,2,\ldots,n.$$

However, the series $\sum_{i=0}^{\infty} u_n(t)$ may be divergent, so we introduce the PHPM to obtain a rapidly convergent series solution.

3. The PHPM for Eq. (1) with (2) and the convergence proof

3.1. Analysis of the piecewise homotopy perturbation method

Divide the interval $[0,b]$ into m subintervals equally, and denote the nodes by x_k, that is, $x_k = kh$, $k = 0,1,\ldots,m$, where $h = b/m$ is the step length. Denote by $u(t)$ the exact solution of Eq. (1) with (2). Let

$$u(t) = u_k(t), \quad t \in [x_{k-1}, x_k], \ k = 1,2,\ldots,m, \tag{10}$$

where $u_k^{(q)}(x_k) = u_{k+1}^{(q)}(x_k)$, $k = 1,2,\ldots,m-1$; $q = 0,1,\ldots,n-1$, where we assume that $u \in C^n[0,b]$.