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Dimensions and bases of hierarchical tensor-product splines

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ABSTRACT

We prove that the dimension of trivariate tensor-product spline space of tri-degree (m, m, m) with maximal order of smoothness over a three-dimensional domain coincides with the number of tensor-product *B*-spline basis functions acting effectively on the domain considered. A domain is required to belong to a certain class. This enables us to show that, for a certain assumption about the configuration of a hierarchical mesh, hierarchical *B*-splines span the spline space.

This paper presents an extension to three-dimensional hierarchical meshes of results proposed recently by Giannelli and Jüttler for two-dimensional hierarchical meshes.

1. Introduction

The spline representations that admit local refinement appeared originally within the framework of geometric design. A new interest in this issue has emerged recently in connection with isogeometric analysis [1]. Let us recall some recent advances in spline representation techniques admitting local refinement.

A well-known T-splines representation defined over two-dimensional T-meshes has been introduced by Sederberg et al. [2]. A practicable mesh refinement algorithm based on the concept of T-splines is described in [3]. The issue of the linear independence of T-splines [4], which is relevant to isogeometric analysis, has been resolved lately by considering analysis-suitable T-splines [5], which are a restricted subset of T-splines. The refinement algorithm for analysis-suitable T-splines is given in [6]. Being an equivalent definition of analysis-suitable T-splines, dual-compatible T-splines [7] are natural candidates for a three-dimensional generalization of analysis-suitable T-splines. However, the refinement algorithm for three-dimensional dual-compatible meshes is still an open problem.

Deng et al. [8] introduced splines over T-meshes. In the case of reduced regularity, with splines of bi-degree (3, 3) and order of smoothness (1, 1), the refinement technique for polynomial splines over hierarchical T-meshes (called PHT-splines) [9] has been given in terms of Bézier ordinates. PHT-splines have been generalized in the form of B-splines for an arbitrary T-mesh [10]. Wu et al. [11] have recently introduced a consistent hierarchical T-mesh. The construction of spline

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Fig. 1. The grid nodes of T' are denoted by black bars. The cells of a domain Ω are denoted by thick solid lines. We note that Ω admits an offset at a distance of 1 but does not admit an offset at a distance of $\frac{3}{2}$.

basis functions has been achieved for a particular type of consistent hierarchical T-meshes [12]. For a three-dimensional hierarchical T-mesh the dimension formula for a spline space of reduced regularity has been derived [13].

Dokken et al. [14] have lately proposed the concept of locally refined splines (LR-splines). LR B-splines are tensor-product B-splines with minimal supports. The refining process is based on so-called hand-in-hand LR-refinement and starts with a tensor-product mesh, which guarantees that the collection of LR B-splines spans the spline space. So far, the refinement algorithm for LR-splines has mostly been developed for the two-dimensional case.

Multilevel B-splines for surface modeling were originally introduced by Forsey and Bartels [15]. Kraft [16] suggested a selection mechanism for hierarchical B-splines that ensures their linear independence as well as local refinement control. In addition, a quasi-interpolant that achieves the optimal local approximation order has been introduced [16]. Vuong et al. [17] looked more clearly at hierarchical B-splines to consider subdomains with partly overlapping boundaries and their applications in isogeometric analysis. Recently, truncated hierarchical B-splines, which are modified to satisfy the partition-of-unity property, have been introduced by Giannelli et al. [18].

In the three-dimensional case, hierarchical B-splines remain the basic approach, allowing a feasible refinement algorithm that guarantees locality of the refinement and linear independence of the blending functions. This paper is inspired by the theoretical results [19] obtained recently by Giannelli and Jüttler for bivariate hierarchical B-splines. For a domain that is a set of cells from an infinite two-dimensional tensor-product grid, it has been proved [19] that the dimension of bivariate tensor-product spline spaces of bi-degree (m, m) with maximal order of smoothness on the domain is equal to the number of tensor-product B-spline basis functions, defined by single knots in both directions, acting effectively on the domain. A reasonable assumption about the configuration of the domain is required. Based on these observations, Giannelli and Jüttler have proved that hierarchical B-splines, produced by Kraft's selection mechanism, span the spline space defined over a hierarchical mesh generated by a decreasingly nested sequence of domains: $\Omega^0 \supset \Omega^1 \supset \cdots \supset \Omega^{N-1}$ associated with an increasingly nested sequence of tensor-product spline spaces $V^0 \subset \cdots \subset V^{N-1}$. Again, a reasonable assumption is required about the configuration of the domains $\Omega^0, \ldots, \Omega^{N-1}$ associated with $V^0, \ldots, V^{N-1}$, respectively.

In this paper, we will prove the analogous results for trivariate hierarchical B-splines on the basis of standard homological algebraic techniques and Mourrain's paper [20]. For a domain that is a set of cells from an infinite three-dimensional tensor-product grid, we will obtain the dimension of the trivariate tensor-product spline space of tri-degree (m, m, m) with maximal order of smoothness on the domain, under the assumption that the domain is homeomorphic to a three-dimensional ball and two-dimensional slices of this domain are simply connected. In addition, we will obtain the dimension of the spline space under the condition that the domain belongs to a certain class (which will be defined in Section 4.1). In this case, we will not impose topological restrictions on the domain itself, apart from restrictions imposed on its configuration with respect to the associated grid. Moreover, we will prove that the dimension of the spline space is equal to the number of tensor-product B-splines, defined by single knots in three directions, acting effectively on a domain of this class.

These results will enable us to prove that hierarchical B-splines, produced by Kraft's selection mechanism [16], span the spline space of tri-degree (m, m, m) with maximal order of smoothness over a three-dimensional hierarchical mesh generated by a decreasingly nested sequence of domains associated with an increasingly nested sequence of tensor-product spline spaces. As in the two-dimensional case, a reasonable assumption about the configuration of the domains will be required. By following our approach, we will also confirm the results obtained by Giannelli and Jüttler [19]. We present our approach gradually, starting from the simplest one-dimensional case.

The rest of this paper is organized as follows: In Sections 2–4 we consider the one-dimensional, two-dimensional, and three-dimensional cases, respectively. Sections 3.1 and 4.1 introduce classes of two-dimensional and three-dimensional domains, respectively, for which we obtain the numbers of tensor-product B-splines acting effectively. In Sections 3.2 and 4.2 we derive the dimension of the tensor-product spline space under certain topological assumptions about two-dimensional and three-dimensional domains, respectively. In addition, we obtain the dimension and a basis of the spline space on a domain of the classes introduced in Sections 3.1 and 4.1. Based on Sections 2–4, we provide in Section 5 a unified proof that B-spline functions, produced by Kraft's selection mechanism [16], span the spline space defined over a hierarchical mesh. Since the final part of the proof is the same as that in [19], we adopt the notation used by Giannelli and Jüttler. We conclude the paper in Section 6. In order to simplify the notation, we will avoid using extra indices related to univariate, bivariate, and trivariate splines, unless otherwise stated. Throughout the paper we will use int Ω to denote the interior of a domain Ω, and supp b to denote the support of a function b.

2. Univariate splines

Let T' be an infinite one-dimensional grid. Without loss of generality, we will suppose that the distances between adjacent grid nodes of T' are equal to 1. A cell of T' is a closed segment of length 1 between adjacent grid nodes.

Let Ω be a closed domain formed by a finite number of cells of T' (see Fig. 1). Then, Ω consists of a number of segments of finite length. A vertex of a domain Ω is a grid node of T' that belongs to Ω. We say that a vertex of Ω is an inner vertex if