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A variational method for multiphase volume-preserving interface motions

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ABSTRACT

We develop a numerical method for realizing mean curvature motion of interfaces separating multiple phases, whose volumes are preserved throughout time. The foundation of the method is a thresholding algorithm of the Bence–Merriman–Osher type. The original algorithm is reformulated in a vector setting, which allows for a natural inclusion of constraints, even in the multiphase case. Moreover, a new method for overcoming the inaccuracy of thresholding methods on non-adaptive grids is designed, since this inaccuracy becomes especially prominent in volume-preserving motions. Formal analysis of the method and numerical tests are presented.

1. Introduction

This work develops a method to compute the area-decreasing evolution of interfaces between an arbitrary number of phases in arbitrary dimension under the constraint that the volume of each phase is preserved throughout time. Such evolutions often appear in situations where interfaces move according to their geometry, while the mass of each phase remains constant (e.g., grain boundaries in ternary alloys, crystal growth, multiphase flows or formation of soap film bubbles). This kind of motion also has applications in image processing (denoising, segmentation), in biology (modeling of vesicles and blood cells), in the description of isolated gravitating systems in general relativity, and other research fields.

Strictly speaking, since volume preservation is a global constraint, one cannot consider a constrained curvature flow directly. Therefore, we have to start from a more basic aspect of the motion, such as the energy. In particular, we consider the constrained steepest descent of the "area energy" of each interface, which counts the measure of interfaces weighted by their corresponding interfacial tensions. The steepest descent of the area energy without any constraint gives the classical mean curvature flow. On the other hand, in the case of two phases, the volume-constrained gradient flow of this energy corresponds to evolution by mean curvature, minus a time-dependent term (equal to the average mean curvature over the

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interface). The situation is analogous for more than two phases but the nonlocal term has a complicated form which depends on the configuration of each interface.

The subject is also mathematically interesting, because it is one of the most simple problems with nontrivial limiting behavior. It is well known that mean curvature flow shrinks uniformly convex smooth hypersurfaces smoothly to a point in finite time. On the other hand, the volume-preserving mean curvature flow converges to the solution of the isoperimetric problem, i.e., a sphere [1–4]. However, the volume-preserving flow may drive general embedded hypersurfaces to self-intersections, as was shown in [5]. On the other hand, [6,7] proved that if the initial surface is sufficiently close to a sphere then it converges to the sphere even if it is not initially convex. Due to the complexity of the multiphase case, there are only a few results concerning the stability of junctions under area-preserving flow, see [8,9] and the references therein.

Since evolution of surfaces is an intensely studied subject of practical interest, a number of analytical and numerical methods have been developed to treat motion by mean curvature. Many of these methods can be applied to the constrained motion addressed here; let us summarize the known results with emphasis on the multiphase case and volume preservation.

Perhaps the most basic approach is to use the definition of the motion itself. That is, in the two-phase case, one computes the evolution of the interface directly from its velocity:

$$\boldsymbol{v}(x) = (-\kappa(x) + \kappa_a)\boldsymbol{n}(x), \quad \text{a.e. } x \in \partial P(t),$$

where $P(t)$ denotes the region occupied by one phase, κ is the mean curvature and κ_a is the average mean curvature over the whole interface [10,11]. These algorithms are effective for computing the evolution of smooth surfaces without topological changes. However, if interaction of different parts of the interface occurs, a complicated decision algorithm is necessary to proceed with the computation, and this becomes increasingly more involved in higher dimensions. Higher dimensions and a larger number of phases also complicate the calculation of curvatures and their averages over the interfaces.

A more general framework is provided by the level set approach which, thanks to its implicit representation of the interface, is able to deal with topological singularities and nonsmooth data. The constrained flow can be realized in this setting by considering the volume-constrained gradient flow of the surface energy functional written in terms of the level set function ϕ,

$$L(\phi) = \int \delta(\phi(x,t))|\nabla\phi(x,t)|\,dx,$$

where δ denotes the Dirac delta function. It is still necessary to calculate the curvature values, but this method can be extended to the multiphase setting by introducing as many level set functions as there are phases and imposing an additional constraint so that the level sets do not overlap or create vacuums (see [12] or [13]). However, such a constraint has an unwanted impact on the flow [14] and the phase volumes are not adequately preserved during the computations. The first problem was solved in [14,15] by employing signed distance functions. Another multiphase modification of the level-set method, which has some similarities to our own approach, was developed in [16], where the constraint of [12] was replaced by a projection step. However, the impact of the projection on the dynamics of the interface was not analyzed.

Another level-set approach, different from our own, and which has received a lot of attention recently is the so-called VIIM (Voronoi Implicit Interface Method) [17]. This method can be used for computing various multiphase physical problems (including motions involving inertial forces, etc.). Although this method overcomes many issues of the level set method, one must still calculate interfacial curvatures and it has yet to be placed within a rigorous framework.

The volume-preserving mean curvature flow arises as a limit of the following nonlocal mass-preserving diffusion equation [18–20]

$$u_t = \Delta u - \frac{1}{\varepsilon^2}W'(u) + \frac{1}{\varepsilon^2|\Omega|}\int_\Omega W'(u)\,dx,$$

where W is a double-well potential and ε is a small parameter related to the width of the diffuse interface. It has been shown that, under suitable conditions, the set

$$P_\varepsilon(t) = \left\{x;\, u_\varepsilon(x,t) \geq \frac{1}{2}\right\}$$

approximates $P(t)$ with error $O(\varepsilon^2|\log\varepsilon|^2)$. Based on this fact, the so-called phase field methods represent interfaces by thin layers in the solution and thus the resolution of this internal layer requires a very fine mesh. On the other hand, this approach handles topological changes without trouble and does not require explicit computation of curvatures (see also [21]).

In this paper we use another approach, often referred to as a thresholding method. We adapt the so-called BMO algorithm from [22] to generate multiphase volume-preserving motion. The BMO algorithm exploits the fact that short-time diffusion of the characteristic function of a region enclosed by an interface (i.e., its convolution with the Gaussian kernel), evolves the interface according to its mean curvature. More precisely, the characteristic function of a region is evolved for a short time by the heat equation and then a thresholding step is carried out to obtain the new interface (given by the 1/2-level set of the diffused function). The main advantage of this approach is that it naturally treats topological changes, produces no intercalary regions and does not require explicit computation of curvatures. Moreover, it is numerically attractive because of its stability and low computational complexity.