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Lie group analysis of nonlinear inviscid flows with a free surface under gravity

Mina B. Abd-el-Malek [a,*,1], Amr M. Amin [b]

[a] Department of Mathematics and Actuarial Science, The American University in Cairo, New Cairo 11835, Egypt

[b] Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt

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ABSTRACT

We applied the Lie group method in studying nonlinear inviscid flows with a free surface under gravity. This method reduces the number of independent variables by one. Therefore, for a system of partial differential equations with three independent variables we applied the method twice to yield a system represented by ordinary differential equations with appropriate corresponding conditions. We obtained analytical solutions for this system. Solutions for the free surface and the velocity components are obtained in closed form. The results are illustrated graphically for different parameters.

1. Introduction

The problem of nonlinear inviscid flows with a free surface under gravity has been investigated in cases for two layers [1], over a cylinder [2], and a rotating channel [3]. Whitham classified many phenomena simulated in atmospheric and oceanic models as *wave-like* flows [4]. These flows are of primary interest in geophysical fluid dynamics, such as gravity waves and Rossby waves. Durran showed that acoustic waves (sound waves) also propagate through all geophysical fluids [5], but in many applications these are small-amplitude perturbations whose detailed structure is of no interest. There have only been a few theoretical studies of the nonlinear and dispersive propagation of water waves with a free surface [6–10]. Freeman found a class of exact solutions for a free surface under the assumption that the flow is simple or involves traveling waves in the horizontal direction [6]. Longuet-Higgins and Cokelet were the first to apply a numerical treatment to full nonlinear water waves [7,8] Ṡachdev and Reddy found some exact solutions describing unsteady-plane gas flows with shocks [10]. Sachdev and Varughese applied the method of infinitesimal transformations to discover new exact solutions for partial differential equations describing free surface flows under gravity [11].

Here we apply the Lie group method, which is a powerful and fundamental tool that provides invariant solutions to boundary value problems. It is applicable to both linear and nonlinear differential equations. The mathematical technique we use is one-parameter group transformation. The Lie group method expresses the infinitesimals of a group in terms of one or more functions, called infinitesimal functions, each of which depends on independent and dependent variables. The procedure for finding the infinitesimals is then reduced to finding the auxiliary equation. This can be obtained by solving a set of partial equations that arise as a result of invoking invariance of the partial differential equations and its

* Corresponding author.
E-mail address: minab@aucegypt.edu (M.B. Abd-el-Malek).

1 On leave from the Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt.

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auxiliary conditions, as previously described [12–14]. Hence, the Lie group method can be used to solve a wider variety of nonlinear problems. There are also several solution techniques for dealing with the determining equations in Lie group analysis of differential equations. Abd-el-Malek and Hassan studied the problem of fission in nuclear fuel [15]. Bira and Sekhar applied the Lie group method to solve modified shallow-water equations [16]. Mabrouk et al. investigated the Lax pair for a generalized Hirota–Satsuma equation [17]. Mekheimer et al. studied an electrically conducting Jeffrey fluid [18] and a hydro-magnetic Maxwell fluid in a porous medium [19]. Mhlanga and Khalique studied generalized Boussinesq–Burgers equations [20]. Parmar and Timol applied a group theoretic approach to natural heat convection and mass transfer for an inclined surface [21]. Özer and Antar investigated two-layer shallow-water equations [22] and Sekhar and Bira applied group analysis to equations for axisymmetric flow of shallow water [23]. Exact solutions for systems of nonlinear partial differential equations are of great interest; such solutions play a major role in the design, analysis, and testing of numerical methods for solving special initial and boundary value problems [16].

Results for the Lie group method are in a good agreement with those obtained using the characteristic function method applied by Abd-el-Malek and Helal [24].

2. Mathematical formulation of the problem

The unsteady two-dimensional inviscid flow equations in the flow domain are the mass conservation equation

$$u_x + v_y = 0 \tag{2.1}$$

and the momentum equations

$$u_t + uu_x + vu_y = -p_x \tag{2.2}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y - g, \tag{2.3}$$

where the Cartesian coordinates x and y are measured along and perpendicular to the uniform horizontal bottom, and u and v are the horizontal and vertical components of the velocity, respectively, as illustrated in Fig. 1. The incompressible fluid has density ρ and pressure p; g denotes acceleration due to gravity.

The flow is bounded from below by the horizontal bottom $y = 0$ and from above by the free surface $y = h(x, t)$. The boundary conditions are

$$v = 0 \quad \text{on } y = 0, \tag{2.4}$$

$$h_t + uh_x - v = 0 \quad \text{on } y = h(x, t). \tag{2.5}$$

Writing

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y},$$

then Eq. (2.3) is reduced to

$$\frac{Dv}{Dt} = -\frac{1}{\rho}p_y - g, \tag{2.6}$$

where $\frac{Dv}{Dt}$ is the total derivative for small v and the vertical acceleration is negligible. Then (2.6) takes the form

$$p_y + \rho g = 0. \tag{2.7}$$

On the free surface $y = h(x, t)$, the pressure is assumed to be constant (equal to p_0, say). Integrating (2.7) for water ($\rho = 1$), we obtain

$$p = p_0 + g(h - y). \tag{2.8}$$

Elimination of p from (2.2) using (2.8) yields

$$u_t + uu_x + vu_y = -gh_x. \tag{2.9}$$

Therefore, the governing equations of motion are

$$u_x + v_y = 0 \tag{2.10}$$

$$u_t + uu_x + vu_y + gh_x = 0. \tag{2.11}$$