Journal of Computational and Applied Mathematics 258 (2014) 57–66

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Efficient evaluation of oscillatory Bessel Hilbert transforms ☆

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ARTICLE INFO

Article history:
Received 12 April 2013
Received in revised form 7 July 2013

MSC:
65D32
65D30

Keywords:
Oscillatory Bessel Hilbert transform
Filon-type method
Gauss–Laguerre quadrature
Adaptive Clenshaw–Curtis quadrature

ABSTRACT

In this paper, we study the asymptotics and evaluation of the oscillatory Bessel Hilbert transform $⨍_0^\infty \frac{f(x)}{x-\tau} J_\nu(\omega x)dx$ with $0 < \tau < \infty$. The singularity of the Hilbert transform is transferred to an individual oscillatory integral independent of $f(x)$. For this singular integral, we present two methods. One is the combination of a Filon-type method and a complex integration method, the other is the combination of a Filon-type method and an adaptive Clenshaw–Curtis quadrature. The remaining integral which is nonsingular can be well calculated with a combination of a Filon-type method and a Gauss–Laguerre quadrature. The efficiency and accuracy of the proposed methods are illustrated by numerical examples.

1. Introduction

The fast computation of the oscillatory Bessel Hilbert transform

$$H^+(f(x)J_\nu(\omega x))(\tau) = ⨍_0^\infty \frac{f(x)}{x-\tau} J_\nu(\omega x)dx, \tag{1.1}$$

where the bar indicates that the integral is the Cauchy principal value at $x = \tau$, plays an important role in many areas of science and engineering, such as astronomy, optics, quantum mechanics, seismology image processing, electromagnetic scattering. For example, in the three-dimensional water-wave radiation problem [1], the best-known method for treating the water-wave radiation problem is to solve an integral equation of the second kind over the wet surface, which involves solving a null-field equation numerically. The null-field equation can be described as follows:

$$\int_{\partial D} \left\{ \phi(q) \frac{\partial}{\partial n_q} \Phi_{jm}^\sigma(q) - V(q)\Phi_{jm}^\sigma(q) \right\} ds_q = 0 \quad (\sigma = 1, 2;\ m, j = 0, 1, 2 \ldots). \tag{1.2}$$

The efficient numerical evaluation of the null-field equation involves the fast computation of oscillatory Bessel Hilbert transforms of the following form:

$$\Phi_n^m(\rho, y) = ⨍_0^\infty k^n e^{-ky} J_m(k\rho) \frac{dk}{k-K}, \tag{1.3}$$

where $\rho > 0$. For more details, one can refer to [1].

☆ This paper was supported by NSF of China (No. 11071260).
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http://dx.doi.org/10.1016/j.cam.2013.08.031

In most of the cases, the oscillatory integral with Bessel kernel cannot be done analytically and one has to resort to numerical methods. Particularly, for large ω, the integrand becomes highly oscillatory. Hence, it presents serious difficulties in obtaining numerical convergence of the integration. In the last few years, many efficient methods have been devised, such as Levin's collocation method [2–4], the modified Clenshaw–Curtis method [5], the Filon-type method [6], the Clenshaw–Curtis–Filon-type method [7], and the generalized quadrature rule [8–10]. Recently, Chen [11,12] rewrote the Bessel function in other forms, and then transformed the integrals into the forms on $[0, \infty]$ that the integrand does not oscillate and decays exponentially fast, which can be efficiently computed by using the Gauss–Laguerre quadrature. However, these methods cannot be applied to evaluating the integral (1.1) directly.

The oscillatory Hilbert transforms with Fourier type oscillator

$$H^{+}(f(t)e^{i\omega t})(x) = ⨍_{\Gamma} e^{i\omega t}\frac{f(t)}{t-x}dt, \quad x > 0,\ x \in \Gamma,$$

have also received considerable attention, where, Γ is an oriented curve in the complex plane. For the case $\Gamma = [-1, 1]$, lots of efficient quadrature methods have been developed (see, for example, [13–18]). For the case $\Gamma = [0, \infty)$, we refer the reader to [19] for a more general overview.

The Hilbert transform (1.1) can be rewritten as

$$H^{+}(f(x)J_\nu(\omega x))(\tau) = \int_0^\infty \frac{f(x)-f(\tau)}{x-\tau}J_\nu(\omega x)dx + f(\tau)⨍_0^\infty \frac{J_\nu(\omega x)}{x-\tau}dx, \tag{1.4}$$

where $\frac{f(x)-f(\tau)}{x-\tau}$ is continuously differentiable if $f \in C^1[0, \infty)$.

Usually, we rewrite $⨍_0^\infty \frac{J_\nu(\omega x)}{x-\tau}dx$ as

$$⨍_0^\infty \frac{J_\nu(\omega x)}{x-\tau}dx = \int_0^a \frac{J_\nu(\omega x)}{x-\tau}dx + ⨍_a^\infty \frac{J_\nu(\omega x)}{x-\tau}dx, \tag{1.5}$$

for some positive constant a with $0 < a < \tau$. The first integral can be efficiently calculated by the Filon-type method [6] and the second can be represented by the integral whose integrand decays exponentially fast.

Especially, for $\nu = 0$ and $\nu = 1$, Wang, Zhang and Huybrechs [19] showed that

$$⨍_0^\infty \frac{J_\nu(\omega x)}{x-\tau}dx = (-1)^{\nu+1}\frac{\pi}{2}[\mathbf{H}_{-\nu}(\omega\tau) + (-1)^\nu Y_\nu(\omega\tau)], \quad \nu = 0, 1,$$

where $\mathbf{H}_{-\nu}(z)$ is the Struve function.

In this paper, we are concerned with efficient methods for computation of the oscillatory Bessel Hilbert transform (1.1). In Section 2, we study the asymptotics of the Bessel Hilbert transform. In Section 3, we present a fast computation of the oscillatory Bessel Hilbert transform. In Section 4, we give several numerical examples to illustrate the effectiveness of the presented methods.

2. Asymptotics of the Bessel Hilbert transform

In this section, we study the asymptotics of the Bessel Hilbert transform (1.1), which provide essential insights into the behavior of the oscillatory Bessel Hilbert transform for large ω. Throughout the paper, we do not distinguish the different constant C.

Theorem 2.1. *For every function $f(x) \in C^1[0, \infty)$, the fixed τ and large values of ω, if $\frac{f(x)-f(\tau)}{x-\tau}$ is monotone and uniformly bounded for $x \geq C$ for some positive constant C, or $\int_1^\infty |f(x)|x^{-\frac{3}{2}}dx < \infty$, then the following holds:*

$$H^{+}(f(x)J_\nu(\omega x))(\tau) = O(\omega^{-1/2}).$$

Before proving Theorem 2.1, we introduce some lemmas.

Lemma 2.1. *For fixed τ and large values of ω, there holds*

$$⨍_a^\infty \frac{J_\nu(\omega x)}{x-\tau}dx = O(\omega^{-1/2}). \tag{2.1}$$

Proof. From [20]

$$i\pi J_\nu(z) = e^{-\frac{\nu}{2}\pi i}K_\nu(ze^{-\frac{1}{2}\pi i}) - e^{\frac{\nu}{2}\pi i}K_\nu(ze^{\frac{1}{2}\pi i}), \quad |\arg z| \leq \frac{\pi}{2},$$

it follows

$$⨍_a^\infty \frac{J_\nu(\omega x)}{x-\tau}dx = \frac{1}{i\pi}\left[e^{-\frac{\nu}{2}\pi i}⨍_a^\infty \frac{K_\nu(-i\omega x)}{x-\tau}dx - e^{\frac{\nu}{2}\pi i}⨍_a^\infty \frac{K_\nu(i\omega x)}{x-\tau}dx\right]. \tag{2.2}$$