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# An efficient fourth-order low dispersive finite difference scheme for a 2-D acoustic wave equation

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ABSTRACT

In this paper, we propose an efficient fourth-order compact finite difference scheme with low numerical dispersion to solve the two-dimensional acoustic wave equation. Combined with the alternating direction implicit (ADI) technique and Padé approximation, the standard second-order finite difference scheme can be improved to fourth-order and solved as a sequence of one-dimensional problems with high computational efficiency. However such compact higher-order methods suffer from high numerical dispersion. To suppress numerical dispersion, the compact and non-compact stages are interlinked to produce a hybrid scheme, in which the compact stage is based on Padé approximation in both $y$ and temporal dimensions while the non-compact stage is based on Padé approximation in $y$ dimension only. Stability analysis shows that the new scheme is conditionally stable and superior to some existing methods in terms of the Courant–Friedrichs–Lewy (CFL) condition. The dispersion analysis shows that the new scheme has lower numerical dispersion in comparison to the existing compact ADI scheme and the higher-order locally one-dimensional (LOD) scheme. Three numerical examples are solved to demonstrate the accuracy and efficiency of the new method.



## 1. Introduction

Numerical solutions of the acoustic wave equations have been widely used in many areas, for example in geophysics, for the purpose of oil exploration, and in medical science, for medical imaging. Among the various numerical methods available, finite difference schemes have attracted the interests of many researchers working on seismic wave propagation (see [1–4] and references therein) and inverse problems [5,6]. Explicit time-stepping finite difference schemes are very popular in practice due to their ease of implementation [7,1,3], but they usually suffer from moderate to severe stability conditions which only allow very small time step size. The implicit finite difference schemes are more complicated and less efficient in terms of implementation. However such schemes are more stable and allow the use of larger time step size, consequently, are more efficient. Another issue with the numerical simulation of acoustic wave propagation is the numerical dispersion. One remedy is to use highly accurate numerical methods. However it is known that the development of numerical methods with good stability and high accuracy remains a challenging task.

Recently, a great deal of efforts have been devoted to developing higher-order finite difference schemes with low numerical dispersion for the above mentioned acoustic equations and elastic wave equations. For example, in [2], using a plane wave theory and the Taylor series expansion, Liu and Sen proposed a new low dispersive time–space domain finite

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difference scheme with error of $O(\tau^2 + h^{2M})$ for 1-D, 2-D and 3-D acoustic wave equations, where $\tau$ and $h$ are time step and spatial grid size, respectively, if a $(2M + 1)$-point stencil is used for each spatial dimension. It was then shown[3] that, along certain fixed directions the error can be improved to $O(\tau^{2M} + h^{2M})$. In [7], Cohen and Joly extended the works of Dablain [8], Shubin and Bell [9] and Bayliss et al. [10] developed a fourth-order accurate explicit scheme with error of $O(\tau^4 + h^4)$ to solve the heterogeneous acoustic wave equation. Chu and Stoffa [11] proposed a new implicit finite-difference method which combines a three-level implicit splitting time integration method in time and implicit finite-difference operators of arbitrary order in space. In [12], Yang etc. developed an optimal nearly-analytic discrete to solve the wave equation in 3D anisotropic media. More related works on the accurate and low dispersive numerical methods can be found in [13–21] and the references therein.

These high order methods are accurate but result in non-compact schemes, which give rise to two issues: efficiency and difficulty in boundary condition treatment. To resolve these issues, many researchers have developed a variety of compact higher-order finite difference schemes to approximate the spatial derivatives. In [22], the authors developed a family of fourth-order three-point combined difference schemes to approximate the first- and second-order spatial derivatives. In [23], the authors introduced a family of three-level implicit finite difference schemes which incorporate the locally one-dimensional method. For more recent compact higher-order difference methods, the readers are referred to [24–28].

The Padé approximation based compact higher-order difference scheme works flawlessly for 1D problems, however, for multi-dimensional problems, the difference operator is replaced by a rational function of the finite difference operator, so a block tridiagonal system needs to be solved at each step. To efficiently solve such a large linear system, some operator splitting techniques are required to provide efficient boundary treatment and break the multi-dimensional problem down to a number of decoupled one-dimensional problems. Two important operator splitting methods: the alternating direction implicit (ADI) method and the locally one-dimensional (LOD) method are widely used for this purpose. The ADI method, which was originally introduced by Peaceman and Rachford [29] to solve parabolic and elliptic equations, has witnessed a lot of development over the years for hyperbolic equations as well [30–32]. For example, Fairweather and Mitchell [33] developed a fourth-order compact ADI scheme (THOC-ADI) for solving the wave equation. Recently, locally one-dimensional (LOD) methods have also been found to be very efficient in solving wave equation. Zhang et al. [34] have developed a fourth-order compact LOD scheme (HOC-LOD) which has lower dispersion than the typical higher-order compact ADI scheme (THOC-ADI), however it uses an additional intermediate variable although the increase in computational cost and computer memory is marginal.

In this paper, we aim to develop a fourth-order finite difference scheme by integrating Padé approximation in temporal and one spatial dimension and the ADI technique. The resulting scheme is fourth-order in both time and space, and this feature is a perfect fit for hyperbolic-type PDEs such as wave equations, since the CFL condition normally requires that the time step size should be proportional to the spatial step size. Another feature of the new scheme is the flexibility which allows non-compact higher-order approximation of the spatial derivative $\frac{\partial^2 u}{\partial x^2}$ to reduce numerical dispersion.

The rest of the paper is organized as follows. A brief introduction of two existing higher-order splitting schemes is presented in Section 2. A new ADI-based compact finite difference scheme is developed, which is then modified to obtain the new efficient low dispersive scheme in Section 3. The numerical dispersion analysis and stability analysis of the new scheme are conducted in Sections 4 and 5, respectively. We then demonstrate the accuracy and efficiency of the new scheme by applying it to solve three numerical examples in Section 6. Finally, the conclusions and possible future extensions are addressed in Section 7.

## 2. Some existing higher-order splitting schemes

In this section we briefly introduce two existing fourth-order compact schemes that have been proposed to solve the 2-D acoustic wave equation in homogeneous media given by

$$\frac{1}{\nu^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x, y) \in \Omega \times [0, T], \tag{1}$$

where $\nu$ is the wave velocity. The initial conditions are

$$u(x, y, 0) = f_1(x, y), \quad (x, y) \in \Omega, \tag{2}$$

$$u_t(x, y, 0) = f_2(x, y), \quad (x, y) \in \Omega \tag{3}$$

and the boundary condition is defined as

$$u(x, y, t) = g(x, y, t), \quad (x, y, t) \in \partial\Omega \times [0, T]. \tag{4}$$

To simplify the discussion, we assume that $\Omega = [0, 1] \times [0, 1]$, which is divided into a uniform $N_x \times N_y$ grid with equal spatial grid spacing $h_x = 1/(N_x - 1)$, $h_y = 1/(N_y - 1)$, and the grid points are defined as $(x_i, y_j) = ((i - 1)h_x, (j - 1)h_y)$ for $1 \le i \le N_x$, $1 \le j \le N_y$. $\tau$ denotes the time step size, while $u^n_{i,j}$ denotes the numerical approximation of $u(x_i, y_j, t_n)$. The following acronyms will be used throughout the rest of the paper to simplify the presentation.