Journal of Computational and Applied Mathematics 252 (2013) 3–11

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Non-symbolic algorithms for the inversion of tridiagonal matrices

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ARTICLE INFO

Article history:
Received 20 September 2011
Received in revised form 8 February 2012

MSC:
15A09
15A15
15A29
39A06
65F05
65Y20

Keywords:
Computational complexity
Difference equation
Inverse matrix
Numerical algorithm
Tridiagonal matrix

ABSTRACT

A representation for the entries of the inverse of general tridiagonal matrices is based on the determinants of their principal submatrices. It enables us to introduce, through the linear recurrence relations satisfied by such determinants, a simple algorithm for the entries of the inverse of any tridiagonal nonsingular matrix, reduced as well as unreduced. The numerical approach is preserved here, without invoking the symbolic computation. For tridiagonal diagonally dominant matrices, a scaling transformation on the recurrences allows us to give another algorithm to avoid overflow and underflow.

1. Introduction

Let $T = \{a_i, b_i, c_i\}$ $(1 \leq i \leq n)$ be an $n \times n$ tridiagonal nonsingular matrix, with $a_1 = c_n = 0$, where the $\{b_i\}$ are the coefficients of the principal diagonal and the $\{a_i\}$, $\{c_i\}$ are those of the lower and upper subdiagonals, respectively. Algorithms for the inversion of such matrices are frequently used. There are efficient packages for the numerical inversion of matrices based on Gaussian algorithms, with pivoting strategies, and for solving linear systems by using the Neville elimination, see e.g. [1]. But they require a great amount of memory and greater run times than other specific algorithms. Concerning the abundant literature about such simpler algorithms for the inversion of tridiagonal matrices, we can refer to [2–4], for example. In general, these specialized algorithms are applicable only in the case of tridiagonal unreduced matrices. Frequently, reduced matrices have been avoided because if an entry on the subdiagonals is null, then the routine can be applied in separate blocks. Indeed, just consider a scenario of tridiagonal strongly reduced matrices, which have numerous null entries in the subdiagonals. Therefore, complexity of such a method of inversion becomes significant. Numerical techniques have also been applied on linear systems with block tridiagonal matrices, see e.g. [5,6].

A first complete analysis on the inversion of tridiagonal nonsingular matrices, without imposing any condition on the coefficients, was introduced in [7]. Nevertheless, the resulting numerical algorithm breaks down when any (left or right) principal submatrix is singular. The symbolic computation recently established in this subject, see e.g. [8,9], overcomes

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2012.05.003

difficulties by considering symbolic parameters, which are adequately replaced in a posterior step of the algorithm. The computational complexity of the algorithms given in [7–9] is $O(n^2)$.

In the applied domain, the numerical approach is currently more spread out and usable than the symbolic one. Thus we try to go on with the numerical line from [7], by introducing a simple algorithm to obtain the entries of the inverse of any tridiagonal nonsingular matrix. There are some compact representations for the entries of the inverse of tridiagonal nonsingular matrices, special as well as general, see e.g. [10,11]. We propose a numerical algorithm by taking advantage of the representation based on the determinants of proper principal submatrices, see e.g. [9],

$$(T^{-1})_{ij} = \begin{cases} (-1)^{i+j}\left(\prod_{k=j+1}^{i} a_k\right)\dfrac{\det T_{j-1}\cdot\det T_{n-i}^{(i)}}{\det T} & \text{if } i > j,\\ (-1)^{i+j}\left(\prod_{k=i}^{j-1} c_k\right)\dfrac{\det T_{i-1}\cdot\det T_{n-j}^{(j)}}{\det T} & \text{if } i \le j. \end{cases} \tag{1}$$

The submatrix T_{i-1} is the left principal one of order $i-1$. The submatrix $T_{n-j}^{(j)}$ is the right principal one of order $n-j$, which begins in the $(j+1)$-th row and column and finishes in the n-th row and column. We define here $\det T_0 = \det T_0^{(n)} = 1$. Representation (1) for the entries of the inverse of tridiagonal nonsingular matrices is a particular case of the closed representation for inverses of nonsingular Hessenberg matrices, see e.g. [12]. We can also obtain Expression (1) by using the companion decomposition, recently introduced in [13], on any tridiagonal nonsingular matrix T.

If in addition T is a symmetric matrix, then its inverse matrix is also a symmetric one, and its entries have the simpler representation,

$$(T^{-1})_{ij} = (T^{-1})_{ji} = (-1)^{M+m}\left(\prod_{k=m+1}^{M} a_k\right)\frac{\det T_{m-1}\cdot\det T_{n-M}^{(M)}}{\det T}, \tag{2}$$

with $M = \max\{i,j\}$, $m = \min\{i,j\}$.

The complexity for the inversion of tridiagonal nonsingular matrices is related to the obtainment of the determinants of all their principal submatrices. For a fast computation we have at our disposal the second order linear difference equations satisfied by such determinants; see also [14]. The linear recurrence relation for determinants of the left principal submatrices, with initial conditions $\det T_1 = b_1$, $\det T_2 = b_2 b_1 - a_2 c_1$, is

$$\det T_{k+2} = b_{k+2}\det T_{k+1} - a_{k+2}c_{k+1}\det T_k, \qquad (1 \le k \le n-2). \tag{3}$$

For determinants of the right principal submatrices, the recurrence relation for $1 \le k \le n-2$, with initial conditions $\det T_1^{(n-1)} = b_n$, $\det T_2^{(n-2)} = b_{n-1}b_n - c_{n-1}a_n$, is

$$\det T_{k+2}^{(n-k-2)} = b_{n-k-1}\det T_{k+1}^{(n-k-1)} - c_{n-k-1}a_{n-k}\det T_k^{(n-k)}. \tag{4}$$

Just consider as we can directly obtain a particular entry of the inverse with $O(n)$ complexity. Although, overflow or underflow can appear in further computation of such recurrences. Thus, our algorithm works for values of the recurrences into the usage range. For example, in some diagonally dominant matrices, i.e. $|b_i| \ge |a_i|+|c_i|$, the solutions of the recurrences grow (or reduce) quickly in magnitude. Therefore other methods should be introduced, such as scaling transformations on the recurrences. We handle these difficulties by considering another algorithm.

The material of this paper is organized as follows. In Section 2, after analyzing difficulties of some current specialized numerical algorithms for the inversion of tridiagonal matrices, [7,9], we point out the features of the algorithm detailed in Appendix A. This algorithm permits us to compute the inverse of any tridiagonal nonsingular matrix of finite order. As an illustration, graphical comparisons of its run times with respect to the built-in function *inv*() of the *Matlab* ® package are given in Fig. 1. In Section 2.2 we check the algorithm of Appendix A on some current examples of tridiagonal matrices. As it was pointed out previously, some difficulties related to overflow and underflow appear in the inversion of various tridiagonal diagonally dominant matrices. We manage these difficulties by introducing in Section 2.3 scaling transformations in the linear recurrences involved. Therefore, an equivalent recursive algorithm is detailed in Appendix B. It permits us to avoid overflow and underflow. To check the complexity of the proposed algorithm from Appendix B with respect to those given in [7,9], a graphical comparison for the mean elapsed time in the inversion of some diagonally dominant matrices is finally provided.

2. Inversion of general tridiagonal matrices

2.1. Algorithms of inversion in the general case

An advance on specialized numerical algorithms for the inversion of general tridiagonal matrices was provided in [7], beyond the classical method using four vectors on unreduced matrices; see e.g. [9, Section 4.1]. This algorithm permits us to