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On new algorithms for inverting Hessenberg matrices

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ABSTRACT

A modification of the Ikebe algorithm for computing the lower half of the inverse of an (unreduced) upper Hessenberg matrix, extended to compute the entries of the superdiagonal, is considered in this paper. It enables us to compute the inverse of a quasiseparable Hessenberg matrix in $O(n^2)$ times. A new factorization expressing the inverse of a nonsingular Hessenberg matrix as a product of two suitable matrices is obtained. Because this allows us the use of back substitution for the inversion of triangular matrices, the inverse is computed with complexity $O(n^3)$. Some comparisons with results obtained using other recent inversion algorithms are also provided.

1. Introduction

The important role of Hessenberg matrices in computational and applied mathematics is well known. In particular they arise in numerical linear algebra, as a result of the application of Givens or Householder orthogonal transformations to a general matrix, when solving the eigenvalue problem, [1,2]. Furthermore the search for fast and simple algorithms for the inversion of such structured matrices is of current interest. The Ikebe algorithm, [3], yields the entries of the upper half of the inverse of any (unreduced) lower Hessenberg matrix with complexity $O(n^2)$. This algorithm provides all the entries of the inverse if the involved matrix is a tridiagonal one. Currently used algorithms utilized specifically for the inversion of tridiagonal matrices are considered in [4].

Two algorithms with complexity $O(n^3)$ have been recently introduced, [5,6], for computing the inverse matrix and the determinant of any (unreduced) nonsingular lower Hessenberg matrix; i.e. with superdiagonal entries $h_{i,i+1} \neq 0$, $(i = 1, 2, \ldots, n-1)$. The method provided in [6] is simpler. Although the procedure described in [5] has a minor flop count, the algorithm given there is more complex because of the way it achieves the inversion of the expanded triangular matrix.

Results about the existence of representations of the inverses of Hessenberg matrices as rank one perturbations of triangular matrices, i.e. in the form $\mathbf{H}^{-1} = \mathbf{T} + \vec{u} \cdot \vec{v}^T$, are known; see e.g. [7,8]. Here the matrix $\mathbf{T}$ represents a particular

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triangular matrix. A constructive example of such a representation for the inverse of a nonsingular lower Hessenberg matrix is given in Theorem 1 of [5].

The aim of this paper is to propose a new characterization of the nonsingular Hessenberg matrices through the factorization

$$\mathbf{H}^{-1} = \mathbf{H}_L \cdot \mathbf{U}^{-1}, \tag{1}$$

for their inverses. A constructive procedure for computing such factorization is also provided.

The matrix $\mathbf{H}_L$ is quasiseparable; i.e. rank $(\mathbf{H}_L(i+1:n, 1:i)) \leq 1$, rank $(\mathbf{H}_L(1:i, i+1:n)) \leq 1$, and $i = 1, 2, \ldots, n-1$; see e.g. [9]. The matrix $\mathbf{U}$ is upper triangular, with ones on its main diagonal. Without loss of generality we consider upper Hessenberg matrices. Analogous results can be obtained for the lower Hessenberg case by taking transposes. In addition, we assume that $h_{i+1,i} \neq 0$ $(i = 1, 2, \ldots, n-1)$; i.e. the matrix $\mathbf{H}$ is unreduced. The reduced case can be handled in a similar way by partitioning the matrix in blocks in an appropriate manner.

The nonsingular lower Hessenberg matrix $\mathbf{H}_L$ is obtained directly by a simple extension of the Ikebe algorithm to the entries of the superdiagonal of $\mathbf{H}^{-1}$. The computational complexity of our proposed algorithm for the inversion of unreduced Hessenberg matrices is equivalent to that of back substitution for the entries of $\mathbf{U}^{-1}$, see e.g. [1,2], plus an additional $O(n^2)$ term.

The procedure introduced here can also be used to obtain a factorization

$$\mathbf{H} = \mathbf{U} \cdot \mathbf{H}_U \tag{2}$$

of the original matrix $\mathbf{H}$. The matrix $\mathbf{H}_U = \mathbf{H}_L^{-1}$ is a quasiseparable nonsingular upper Hessenberg matrix.

When $\mathbf{H}$ is also quasiseparable the back substitution stage can be avoided. In this situation, the expanded Ikebe algorithm provides a faster computation of the inverse matrix with complexity $O(n^2)$.

The structure of the paper is as follows. In Section 2, after recalling the Ikebe algorithm, we demonstrate the factorization (1) and show how to compute the inverse matrix $\mathbf{H}^{-1}$ using the algorithm detailed in Appendix A. In Section 3 a customary example and graphical comparisons of the elapsed times are introduced for quasiseparable Hessenberg matrices, and also for matrices associated to the upper Hessenberg form of nonsingular matrices taking on random values in $(-5; 5)$. Some conclusions are outlined at the end.

2. An extension of the Ikebe algorithm for computing the inverses of Hessenberg matrices

2.1. The Ikebe algorithm for the lower half of the inverse of nonsingular Hessenberg matrices

In order to obtain the inverse factorization (1), we begin with the Ikebe algorithm from [3], adapted here to an (unreduced) nonsingular upper Hessenberg matrix of order n. It gives us the lower half of the inverse matrix $\mathbf{H}^{-1}$, i.e. $h_{i,j}^{(-1)}$ with $i \geq j$. Following [3], we have

$$h_{i,j}^{(-1)} = y(i) \cdot x(j); \quad i \geq j, \tag{3}$$

where $y(i)$ and $x(j)$ are the ith and jth components of the vectors $\vec{y}$ and $\vec{x}$, respectively.

The components of the vector $\vec{x}$ were achieved in the following recursive way, with $h_{j,j-1}^{-1} = 1/h_{j,j-1}$,

$$x(1) = \lambda \neq 0 \quad \text{(an arbitrary constant)},$$

$$x(j) = -h_{j,j-1}^{-1} \sum_{k=1}^{j-1} h_{k,j-1} x(k) \quad (j = 2, 3, \ldots, n). \tag{4}$$

The components of the vector $\vec{y}$ were given by the following recurrence,

$$y(n) = \left(\sum_{k=1}^{n} h_{k,n} x(k) \right)^{-1},$$

$$y(i) = -h_{i+1,i}^{-1} \sum_{k=i+1}^{n} h_{i+1,k} y(k) \quad (i = n-1, n-2, \ldots, 1). \tag{5}$$

In addition, we can recover from $y(n)$ the value of $\det \mathbf{H}$, the determinant of $\mathbf{H}$, with the convention $\det \mathbf{H}_0^{(n)} = 1$, by

$$\det \mathbf{H} = (-1)^{n-1} \frac{\left(\prod_{k=2}^{n} h_{k,k-1} \right)}{\lambda \cdot y(n)}. \tag{6}$$

We define now a lower triangular matrix $\mathbf{L}^*$ with its ij entry ($i \geq j$) given by (3). These entries constitute the lower half of $\mathbf{H}^{-1}$.