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Increasing the approximation order of spline quasi-interpolants

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ABSTRACT

In this paper, we show how by a very simple modification of bivariate spline discrete quasi-interpolants, we can construct a new class of quasi-interpolants which have remarkable properties such as high order of regularity and polynomial reproduction. More precisely, given a spline discrete quasi-interpolation operator Q_d, which is exact on the space $\mathbb{P}_m$ of polynomials of total degree at most m, we first propose a general method to determine a new differential quasi-interpolation operator Q_r^D which is exact on $\mathbb{P}_{m+r}$. Q_r^D uses the values of the function to be approximated at the points involved in the linear functional defining Q_d as well as the partial derivatives up to the order r at the same points. From this result, we then construct and study a first order differential quasi-interpolant based on the C^1 cubic B-spline on the equilateral triangulation with a hexagonal support. When the derivatives are not available or extremely expensive to compute, we approximate them by appropriate finite differences to derive new discrete quasi-interpolants $\tilde{Q}_d$. We estimate with small constants the quasi-interpolation errors $f - Q_r^D[f]$ and $f - \tilde{Q}_d[f]$ in the infinity norm. Finally, numerical examples are used to analyze the performance of the method.

1. Introduction

Quasi-interpolation based on a B-spline, i.e. a compactly supported piecewise polynomial function, is a general approach for efficiently constructing approximants, with low computational cost. Its effectiveness is particularly due to its small support to achieve local control via suitable spline coefficients in the space spanned by the translates of the B-spline. It possesses certain very nice approximation properties such as its numerical stability and easy implementation. It also generates a numerical method that avoids the solution of large full linear systems.

The various methods in the literature for producing quasi-interpolants are excellently documented in the comprehensive book [1] (see also [2,3], the survey [4], and the references therein). In [5], it is shown how to modify a given linear operator such that the resulting operator reproduces polynomials to the highest possible degree, and such that the approximation order is the best possible. Also the derivation of error estimates for those quasi-interpolants is well documented (see e.g. [6–10] and the references therein).

The aim of this paper is to develop and apply this method to spline quasi-interpolants to derive new explicit differential spline operators, based on a uniform type-1 triangulation τ approximating regularly distributed data. An important feature

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of the proposed operators is that they only use the values of the function to be approximated as well as their derivatives up to some prescribed order at the grid points. We emphasize that our approach differs from the existing techniques of quasi-interpolation in both theoretical and computational aspects.

The paper is organized as follows. In Section 3, we construct a new differential quasi-interpolation operator Q_r^D (abbr. DQIO) based on a discrete spline quasi-interpolation operator Q_d (dQIO for short) exact on the space $\mathbb{P}_m$ of polynomials of total degree at most m and by using the derivatives of the approximated function up to the order r. We motivate the introduction of these differential operators and sketch their approximation properties. In particular, we establish an integral representation of the associated error and we prove that the resulting operator is exact on $\mathbb{P}_{m+r}$. Moreover, we estimate error bounds in order to define a function to be minimized. In Section 4, we particularize that function to the C^1 cubic case on the equilateral type-1 triangulation, and the unique solution of the associate problem is determined. Based on the dQIO given for that solution and using the values of the gradients, we construct a C^1 quartic differential quasi-interpolant which is exact on $\mathbb{P}_3$, whence the approximation power is achieved. For this setting we establish a more explicit expression for the error estimate. When the derivatives are not available, we use appropriate finite difference schemes to approximate them. Then a discrete C^1 quartic spline quasi-interpolant is derived and the error estimate is obtained in Section 5. Finally, in Section 6, numerical experiments are analyzed to show the performance of the proposed differential and discrete quasi-interpolants, and the latter is compared with another quasi-interpolation scheme based on C^1 quartic splines on a (slightly different) type-1 triangulation given in [11], that does not require any derivative at any point of the domain. We conclude by briefly indicating their advantages, possible extensions of our results and some studies in progress.

2. Notations

Let τ be a uniform triangulation of the plane with grid points $(A_i)_{i\in\mathbb{Z}^2}$. Let us denote by $\mathbb{P}_k$ the space of bivariate polynomials of total degree at most k, and by $S_k^l(\tau)$ the space of piecewise polynomial functions in $C^l\left(\mathbb{R}^2\right)$ of total degree at most k, defined on τ. If $M \in S_k^l(\tau)$ is a B-spline, we denote by $\mathbb{P}(M)$ the space of polynomials of maximal total degree included in the space $\mathcal{S}(M)$ spanned by translates of M. We will assume throughout the paper that $\mathbb{P}(M) = \mathbb{P}_m$ for some positive integer m.

For a real valued function f and $k \in \mathbb{N}$, we say $f \in C^k\left(\mathbb{R}^2\right)$ if f is k times continuously differentiable in the following sense: the directional derivatives of order l, $l = 0, \ldots, k$, at $x \in \mathbb{R}^2$ along the direction $y \in \mathbb{R}^2$ defined as

$$D_y^l f(x) = \frac{d^l}{dt^l} f(x + ty)\,|_{t=0}$$

exist and depend continuously on x. When the directional derivative exists for y, it may be extended to multiples by defining

$$D_{\alpha y}^l f(x) = \alpha^l D_y^l f(x), \quad \alpha \in \mathbb{R}.$$

For $f \in C^k\left(\mathbb{R}^2\right)$, we introduce

$$\left|D^k f\right| = \sup_{x\in\mathbb{R}^2} \sup\left\{\left|D_y^k f(x)\right| : y \in \mathbb{R}^2, \|y\| = 1\right\},$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$. It follows that for any $x, y \in \mathbb{R}^2$, we have

$$\left|D_y^k f(x)\right| \le \left|D^k f\right| \|y\| .$$

To make the distinction clear, the superscript on a differential operator written in the form Q_r^D is meant to remind us that this quasi-interpolant is based on derivatives up to order r, while the notation Q_d is designed for those based on point evaluators.

3. Modified differential bivariate spline quasi-interpolants

In this section we briefly outline the approach we take and state the main result of this paper. For a given non-negative integer ℓ, consider $\ell + 1$ distinct vectors $v_0, \ldots, v_\ell$ with $v_0 = 0$, and $v_k \neq 0$ otherwise. We are interested in the dQIO Q_d based on the B-spline $M \in S_k^l(\tau)$ given by the expression

$$Q_d[f](x) := \sum_{i\in\mathbb{Z}^2} \lambda\left[f(\cdot + A_i)\right] M(x - A_i), \tag{1}$$

where λ is the linear functional defined as

$$\lambda[f] := \sum_{k=0}^{\ell} \sum_{j\in J_k} c_{k,j} f\left(-A_j + v_k\right),$$

for finite subsets $J_0, \ldots, J_\ell \subset \mathbb{Z}^2$ and $c_k := \left(c_{k,j}\right)_{j\in J_k} \in \mathbb{R}^{\#J_k}$, with $\#J$ denoting the cardinality of J. Q_d is a linear map into $\mathcal{S}(M)$ which is local and bounded, and we shall construct Q_d to reproduce $\mathbb{P}_m$.