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A higher order uniformly convergent method with Richardson extrapolation in time for singularly perturbed reaction–diffusion parabolic problems

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ABSTRACT

In this paper, we are interested in solving efficiently an initial-boundary value singularly perturbed time-dependent problem of reaction–diffusion type. On *a priori* special mesh we construct a high order uniformly convergent finite difference scheme which combines the implicit Euler method to discretize in time, together with the Richardson extrapolation technique, and a HODIE scheme to discretize in space. The analysis of the uniform convergence splits completely the contribution to the global error of both the time and the space discretizations. We show numerical results for different test problems confirming in practice the order of uniform convergence proved.

1. Introduction

In this paper, we consider singularly perturbed parabolic initial-boundary value problems in one space dimension of type

$$\begin{cases} L_\varepsilon u \equiv u_t + L_{x,\varepsilon} u = f(x,t), & (x,t) \in Q = \Omega \times (0,T] \equiv (0,1) \times (0,T], \\ u(x,0) = 0, \quad x \in \overline{\Omega}, & u(0,t) = u(1,t) = 0, \quad t \in (0,T], \end{cases} \tag{1}$$

where the reaction–diffusion differential operator $L_{x,\varepsilon}$ is defined by

$$L_{x,\varepsilon} u \equiv -\varepsilon u_{xx} + b(x,t)u. \tag{2}$$

We assume that the diffusion parameter is positive, $0 < \varepsilon \le 1$, and it can be arbitrarily small, that the reaction term is strictly positive, $b(x,t) \ge \beta > 0$ for all $(x,t) \in \overline{Q}$, and also that the data of the problem are sufficiently smooth functions.

It is well-known (see [1–3]) that the solution of (1) has a boundary layer at $x = 0, 1$ of width $\mathcal{O}(\sqrt{\varepsilon}|\ln \varepsilon|)$. Moreover, under sufficient compatibility conditions at the corners $(0,0)$ and $(1,0)$, the solution of (1) satisfies the bounds (see [1] for instance)

$$|u^{(k,m)}(x,t)| \le C(1 + \varepsilon^{-k/2} B_\varepsilon(x)), \quad 0 \le k + 2m \le 8, \tag{3}$$

where the function $B_\varepsilon(x)$ is given by $B_\varepsilon(x) = e^{-\sqrt{\beta/\varepsilon}x} + e^{-\sqrt{\beta/\varepsilon}(1-x)}$. Moreover, the solution can be decomposed in the form $u = \phi + \psi$, where ϕ is the regular component and ψ is the singular component, and they satisfy the bounds

$$|\phi^{(k,m)}(x,t)| \le C(1 + \varepsilon^{2-k/2}), \qquad |\psi^{(k,m)}(x,t)| \le C\varepsilon^{-k/2} B_\varepsilon(x), \quad 0 \le k + 2m \le 8.$$

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The efficient resolution of singularly perturbed time-dependent problems has been an interesting subject in the past years. For instance, in [4–7] numerical methods were developed to solve 1D parabolic problem for both linear or nonlinear singularly perturbed reaction–diffusion problems; in [8–10], the case of 2D parabolic problems was analyzed and singularly linear systems of reaction–diffusion type were solved in [11,12].

In [13] a method combining the implicit Euler method and a HODIE (High Order via Differential Identity Expansion) compact fourth order scheme (see [14]) was used. In that paper, the authors proved that the method has first order uniform convergence in time and almost fourth order uniform convergence in space. The analysis of the convergence was based on a two step discretization process (see [9,5,10]). In the first step, the continuous problem is discretized only in time, resulting in a family of 1D linear stationary singularly perturbed problems depending on the time discretization parameter. In the second step, those problems are discretized in space. The main advantage of this technique is that it permits to analyze independently the contribution to the error of the time and space discretizations. Nevertheless, the method of [13] only gives first order of uniform convergence in time variable. Here we see that it is possible to combine this idea with the use of the Richardson extrapolation technique applied only to the time discretization. Then, the fully discrete method gives second order of uniform convergence in time and almost fourth order in space. So, the improved approximation is second order globally convergent in contrast with the first order proved in [13].

The Richardson extrapolation technique has been used to approximate the solution of singularly perturbed problems, improving the numerical approximation of a basic scheme in the case of 1D steady problems of convection–diffusion type [15], parabolic problems of reaction–diffusion and convection–diffusion type [16–18], and elliptic problems of convection–diffusion and reaction–diffusion type [19,20]. In all that papers, the main difficulties in the analysis of the uniform convergence of the extrapolated approximation were related with the space variable. In this paper, we show that in the case of parabolic problems, the extrapolated approximation for the time variable can be analyzed independently of the space variable.

The paper is structured as follows. In Section 2, we consider the time discretization, recalling the first order uniform convergence of the classical backward Euler method on a uniform mesh, and we prove that the Richardson extrapolation gives second order of uniform convergence. In Section 3, a hybrid finite difference scheme, which uses central difference or a stable compact HODIE scheme of fourth order, is defined to discretize the semidiscrete problems. That hybrid method is constructed on a special mesh of Vulanović type (see [21,22]), and we prove that the fully discrete method has second order convergence in time and almost fourth order convergence in space. Finally, in Section 4 we show some numerical results for different test problems, illustrating in practice the efficiency of the method and corroborating the improvement in the order of uniform convergence of the new scheme with respect to the scheme given in [13].

Henceforth, C denotes a generic positive constant independent of the diffusion parameter ε and also of the discretization parameters N and M.

2. Time semidiscretization: Euler method and Richardson extrapolation

On the uniform mesh $\overline{\omega}^M = \{t_k = k\tau,\ 0 \le k \le M, \tau = T/M\}$, the implicit Euler method is used to discretize the time variable. Then, the semidiscrete problem reads

$$\begin{cases} z(x, t_0) = 0, \\ \begin{cases} (D_t^- + L_{x,\varepsilon})z(x, t_n) = f(x, t_n), & 1 \le n \le M, \\ z(0, t_n) = z(1, t_n) = 0, \end{cases} \end{cases} \tag{4}$$

where the backward time difference is given by

$$D_t^- z(x, t_n) := \frac{z(x, t_n) - z(x, t_{n-1})}{\tau}.$$

It is well-known that the implicit Euler method satisfies $|u(x, t_n) - z(x, t_n)| \le C\tau$, which proves the uniform convergence of the time discretization. Nevertheless, in this paper we write the semidiscrete problem (4) in vector form because we want to deduce also appropriate estimates of the extrapolated solution of the semidiscrete problem using the Richardson extrapolation technique. With this aim, we rewrite problem (4) as follows

$$\mathbf{L}^M \mathbf{z}(x) = \mathbf{f}(x), \quad x \in \Omega, \qquad \mathbf{z}(0) = \mathbf{z}(1) = \mathbf{0}, \tag{5}$$

where

$$\begin{aligned} \mathbf{z}(x) &= (z(x, t_0), z(x, t_1), \ldots, z(x, t_M))^T, \\ \mathbf{f}(x) &= (0, f(x, t_1), f(x, t_2), \ldots, f(x, t_M))^T, \\ \mathbf{L}^M &= (L^{M,0}, L^{M,1}, \ldots, L^{M,M})^T, \end{aligned}$$

and the components of the vector operator $\mathbf{L}^M$ are defined by

$$L^{M,n}\mathbf{z} := \begin{cases} z(x, t_0), & \text{if } n = 0, \\ (D_t^- + L_{x,\varepsilon})z(x, t_n), & \text{if } 1 \le n \le M. \end{cases}$$