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A derivative-free explicit method with order 1.0 for solving stochastic delay differential equations[☆]

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ABSTRACT

A new explicit stochastic scheme of order 1 is proposed for solving stochastic delay differential equations (SDDEs) with sufficiently smooth drift and diffusion coefficients and a scalar Wiener process. The method is derivative-free and is shown to be stable in mean square. A stability theorem for the continuous strong approximation of the solution of a linear test equation by the Milstein method is also proved, which shows the stepsize restriction for stability is larger than those given previously in the literature. The case of linear SDDEs is further investigated, in order to compare the stepsize restrictions for stability of these two methods. Numerical experiments are given to illustrate the obtained stability properties.

1. Introduction

Stochastic delay differential equations (SDDEs) have become an important tool in many scientific areas due to its application for modeling dynamical systems. Some areas where SDDEs are used in modeling include economics, biology and medicine [1–6], to name a few. These models can usually offer a far more realistic representation of the physical system compared to deterministic models without delay since the uncertainty and delay processes are ubiquitous. However, as in the case of deterministic delay differential equations (DDEs) and stochastic differential equations (SDEs) without time delay, only a few, very simple SDDEs can be solved analytically. As a consequence, there is the need for designing numerical methods for approximating their solutions.

Many standard ODE methods have been extended to functional equations with different kinds of memory terms. So there is a rich theory for designing effective numerical methods for solving deterministic DDEs. The research in the numerical analysis for SDEs has also made a lot of advances. An overview of these results can be found in some monographs and survey papers, see for example Higham [7], Saito and Mitsui [8] and Buckwar and Sickenberger [9]. But for stochastic DDEs the numerical analysis is less well-developed.

In [10], Küchler and Platen studied Euler-type and Milstein-type schemes for SDDEs. They proved that these approximations converge with order 0.5 and 1.0 in mean-square, respectively. The mean-square convergence and stability

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of the semi-implicit Euler method for a linear SDDE is considered by Liu, Cao and Fan [11]. Buckwar [12] considered the θ-Maruyama methods for stochastic functional differential equations (SFDES) involving a distributed delay term. A variant of the Euler–Maruyama method is considered for SDDEs with variable delay, see Mao and Sabanis [13]. In [14], Hu, Mohammed and Yan developed a strong Milstein approximation scheme for solving stochastic delay differential equations (SDDEs). They showed that the appropriate version of the Itô formula for the numerical analysis of SFDEs requires the application of Malliavin calculus. Wang and Zhang [15] dealt with the adapted Milstein method for solving linear SDDEs and proved that this numerical method is mean-square (MS) stable under suitable conditions. However, all these Euler-type schemes have order 0.5 in mean-square while higher order numerical methods such as Milstein-type schemes involve considerable complexities in implementation because of the approximation of higher order stochastic integrals and the evaluation of higher order derivatives of both the drift and diffusion coefficients.

Therefore, the application of derivative-free methods for SDDEs is very promising for future research. The main advantage of derivative-free methods is the cheaper numerical approximation. Here, cheap means without additional evaluations of the derivatives of both the drift and diffusion coefficients. Thus, in this paper, we adapted an explicit order 1.0 method that was first presented in [16] to approximate the solutions of SDEs. In Section 2, we will describe the scheme in more detail. It is easier to implement than a Milstein-type method while it has a higher order than Euler-type methods. In Section 3, we investigate the mean-square stability (MS-stability) of the method. A numerical stepsize restriction for stability is derived such that this type of method preserves the MS-stability of the underlying equation. In Section 4, we consider the MS-stability of the Milstein method for linear SDDEs and obtain a stepsize restriction for stability from which a larger stepsize restriction for stability can be derived than previously given in the literature [15]. This shows that the stepsize restriction for stability obtained in [15] is not optimal. We also investigate under what conditions our method has better stability properties than the Milstein method. Section 5 will include some numerical examples to illustrate these theoretical results.

2. The derivative-free explicit order 1.0 strong method

We consider the following Itô-type scalar SDDEs with delay $\tau > 0$:

$$\begin{cases} dX(t) = f(t, X(t), X(t-\tau))dt + g(t, X(t), X(t-\tau))dW(t), \quad t \in [0, T], \\ X(t) = \psi(t), \quad t \in [-\tau, 0], \end{cases} \tag{2.1}$$

where $W(t)$ is a standard Wiener process given on the probability space $(\Omega, \mathscr{A}, P)$ with respect to a filtration $(\mathscr{A}_t)_{t\geq 0}$. The initial condition $\psi(t)$ is an $\mathscr{A}_0$-measurable $C([-\tau, 0]; \mathbb{R})$-valued random variable with $E\|\psi\|^2 < \infty$. The class of SDDEs are used in many areas. For example, authors employed them to describe the dynamics of the transcriptional factor TF-A in mammalian cells in [6]. They were also used to model the over-damped particle motion in the double-well quartic potential [17]. For numerically solving SDDEs (2.1), we take a uniform mesh on $[0, T]$: $0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots < t_N \leq T$, where $t_n = t_0 + nh$, $n = 0, \ldots, N$, $hN \leq T$, $h(N+1) > T$, $N \in \mathbb{N}$. In addition, the choice of h is not arbitrary, it has to be chosen such that $l := \tau/h \in \mathbb{N}$. In other words, the delay period τ has to be a multiple of h. This restriction on the stepsize is very common both when solving Delay Differential Equations and when analyzing the performance of numerical methods. Then the Milstein method for solving (2.1) is

$$\begin{aligned} Y_{n+1} &= Y_n + f(t_n, Y_n, Y_{n-l})h + g(t_n, Y_n, Y_{n-l})\triangle W_n \\ &\quad + g_1'(t_n, Y_n, Y_{n-l})g(t_n, Y_n, Y_{n-l})I_1 + g_2'(t_n, Y_n, Y_{n-l})g(t_{n-l}, Y_{n-l}, Y_{n-2l})I_{11}, \end{aligned} \tag{2.2}$$

where,

$$g_1'(t_n, Y_n, Y_{n-l}) = \frac{\partial g(t_n, Y_n, Y_{n-l})}{\partial Y_n}, \tag{2.3}$$

$$g_2'(t_n, Y_n, Y_{n-l}) = \frac{\partial g(t_n, Y_n, Y_{n-l})}{\partial Y_{n-l}}, \tag{2.4}$$

$$I_1 = \frac{\triangle W_n^2 - h}{2}, \tag{2.5}$$

and

$$I_{11} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dw(u-\tau)dw(s). \tag{2.6}$$

The double stochastic integral I_{11} can be simulated using (B.3) in [14]. From Ref. [16], we know that $\triangle W_n$ and the double integrals I_1,1_{11} satisfy

$$E(\triangle W_n) = E(I_1) = E(I_{11}) = 0; \qquad E((\triangle W_n)^2) = h; \tag{2.7}$$

$$E(I_1^2) = E(I_{11}^2) = h^2/2; \qquad E(\triangle W_n I_1) = E(\triangle W_n I_{11}) = E(I_1 I_{11}) = 0. \tag{2.8}$$