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Generalization of an existence theorem for complementarity problems[☆]

Qingjie Hu[a,*], Yu Chen[a], Jingtong Wang[b], Zi-sheng Ouyang[b]

[a] School of Mathematics and Computing Science, Guilin University of Electronic Technology, 541004, Guilin, China

[b] School of Mathematics and Statistics, Hunan University of Commerce, 410205, Changsha, China

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ABSTRACT

In this paper, we present a new notion of exceptional d-regular mapping, which is a generalization of the notions of exceptional regular mapping and d-regular mapping. By using the new notion, we establish a new existence result for complementarity problems. Our results only generalize Karamardian's and Zhao's existence results (Theorem 3.1 in Karamardian (1972) [5], Theorem 3.8 in Harker et al. (1990) [2], Theorem 4.1 in Zhao and Isac (2000) [6], Theorem 3.1 in Zhao (1999) [13]). In our analysis, the notion of a new generalized exceptional family of elements for complementarity problems plays a key role.

1. Introduction

The complementarity problem (CP) is to find a vector $x \in R^n$ such that

$$x \geq 0, f(x) \geq 0, \quad x^T f(x) = 0, \tag{1.1}$$

where $f : R^n \to R^n$ is a continuous function. Such a problem has been studied extensively by many authors, since it has many wide applications in engineering, economics, operations, etc. [1–3].

Since the existence of a solution to CPs is not always assured, the study of existence conditions is very important in the theory, algorithm development, and applications of CPs. Throughout this paper, our focus is to develop a new existence result for CPs. We first review some previous results. In [4], Moré proved that, if the mapping $f(x)$ is strictly copositive and there exists a mapping $c : R_+ \to R$ such that $c(\lambda) \to \infty$ as $\lambda \to \infty$ and, for all $\lambda \geq 1, x \geq 0, [f(\lambda x) - f(0)]^T x \geq c(\lambda)[f(x) - f(0)]^T x$, then the CP has a nonempty, compact solution set. To relax the strict copositivity assumption, Karamardian [5] proved that, if the mapping $G(x) = f(x) - f(0)$ is positively homogeneous of degree α and $G(x)$ satisfies d-regularity for some $d \in R^n$ and $d > 0$, then CP has a nonempty solution set (Theorem 3.1 in [5]); see also Theorem 3.8 in [2]. In [6], by using the notion of an exceptional family of elements for complementarity problems [7–12], Zhao and Isac showed that, if the mapping $G(x) = f(x) - f(0)$ is positively homogeneous of degree $\alpha = 1$ and $G(x)$ satisfies exceptional regularity, then the CP has a solution (Theorem 4.1 in [6]). Subsequently, in [13], Zhao further showed that if the mapping $G(x) = f(x) - f(0)$ is generalized positively homogeneous and $G(x)$ satisfies exceptional regularity, then the CP has a solution (Theorem 3.1 in [13]).

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* Corresponding author.
E-mail address: hqj0525@126.com.cn (Q. Hu).

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In this paper, motivated by Karamardian [5], Zhao [13,14], and Zhao and Isac [6], by introducing the new notion of exceptional d-regular mapping, which is a generalization of the notions of exceptional regular mapping and d-regular mapping, we establish a new existence result for complementarity problems. Our results generalize Karamardian's and Zhao's existence results (Theorem 3.1 in [5], Theorem 3.8 in [2], Theorem 4.1 in [6], and Theorem 3.1 in [13]). It is worth noting that the notion of a new generalized exceptional family of elements for complementarity problems plays an important role in our analysis.

2. A new generalized exceptional family of elements

To show our main result, we first introduce the notion of a new generalized exceptional family of elements, which plays a key role in the subsequent analysis.

Definition 2.1. Given $d \in R^n_{++}$, we say that $\{x^r\} \subset R^n_+$ is a generalized exceptional family of elements for the function f if $\|x^r\| \to \infty$ as $r \to \infty$, and for each x^r there exist a positive number μ^r and a number $\theta \in [0, 1]$, such that

$$f_i(x^r) = -\mu^r \left((1-\theta)\frac{x_i^r}{\|x^r\|} + \theta d_i \right), \quad \text{if } x_i^r > 0, \tag{2.1}$$

$$f_i(x^r) \geq -\mu^r \theta d_i, \quad \text{if } x_i^r = 0. \tag{2.2}$$

It is well known that the classical variational inequality problem, denoted by $VI(K, f)$, is to find a solution x^* such that

$$(x - x^*)^T f(x^*) \geq 0, \quad \forall x \in K,$$

where K is a closed convex set in R^n. In particular, when the set $K = R^n_+ = \{x \in R^n : x \geq 0\}$, $VI(K, f)$ reduces to the CP.

Let $d \in R^n$ be a given positive vector. For such d, let

$$K_r = R^n_+ \cap \{x \in R^n : (1-\theta)\|x\| + \theta x^T d \leq r\},$$

where r is a positive number and $\theta \in [0, 1]$. Obviously, K_r is a bounded convex set. This shows that $VI(K, f)$ has at least one solution [2].

To get a new alternative theorem, we present the following lemma, which is similar to that in [1,14].

Lemma 2.1. *Let $f : R^n \to R^n$ be a continuous function. Then the CP has a solution if and only if there exist a positive number r and some $\theta \in [0, 1]$ such that $VI(K_r, f)$ has a solution x^r with $(1-\theta)\|x^r\| + \theta (x^r)^T d < r$.*

Proof. If x^* is a solution of the CP, then

$$(x - x^*)^T f(x^*) \geq 0, \quad \forall x \in R^n_+.$$

Let $r > (1-\theta)\|x^*\| + \theta (x^*)^T d$. Obviously, we have

$$(x - x^*)^T f(x^*) \geq 0, \quad \forall x \in K_r.$$

This implies that x^* is a solution of $VI(K_r, f)$.

Conversely, assume that there exist some positive number r and some $\theta \in [0, 1]$ such that $VI(K_r, f)$ has a solution x^r with $(1-\theta)\|x^r\| + \theta (x^r)^T d < r$, i.e.,

$$(x - x^r)^T f(x^r) \geq 0, \quad \forall x^r \in K_r. \tag{2.3}$$

To prove that x^r is a solution of the CP, it is sufficient to show that

$$(x - x^r)^T f(x^r) \geq 0, \quad \forall x^r \in R^n_+ \backslash K_r. \tag{2.4}$$

Actually,

$$p(\lambda) = \lambda x + (1-\lambda) x^r \in R^n_+, \quad \forall x \in R^n_+, \forall \lambda \in [0, 1].$$

Taking into account that $(1-\theta)\|x^r\| + \theta (x^r)^T d < r$, there exists a sufficiently small positive number λ^* such that $(1-\theta)\|p(\lambda^*)\| + \theta p(\lambda^*)^T d < r$; hence $p(\lambda^*) \in K_r$. From (2.3), we have

$$\begin{aligned} 0 &\leq (p(\lambda^*) - x^r)^T f(x^r) = (\lambda^* x + (1-\lambda^*) x^r - x^r)^T f(x^r) \\ &= \lambda^* (x - x^r)^T f(x^r), \end{aligned}$$

which implies that (2.4) holds. So, x^r is a solution of the CP. □