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A fully discrete C^0 interior penalty Galerkin approximation of the extended Fisher–Kolmogorov equation

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ABSTRACT

A fully discrete C^0 interior penalty finite element method is proposed and analyzed for the Extended Fisher–Kolmogorov (EFK) equation $u_t + \gamma \Delta^2 u - \Delta u + u^3 - u = 0$ with appropriate initial and boundary conditions, where γ is a positive constant. We derive a regularity estimate for the solution u of the EFK equation that is explicit in γ and as a consequence we derive *a priori* error estimates that are robust in γ.

1. Introduction

We study a fully discrete C^0 interior penalty method for the fourth order parabolic Extended Fisher–Kolmogorov (EFK) equation:

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u - \Delta u + \phi(u) = 0 \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \gamma \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T], \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \tag{1.3}$$

where $\Omega \in \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded domain with convex polyhedral boundary $\partial\Omega$, $T > 0$, γ is a positive constant, u_0 is a given function of $x \in \mathbb{R}^d$ and $\phi(u) = u^3 - u$. The nonlinear function $\phi(u) = \Psi'(u)$ for $\Psi(u) = \frac{1}{4}(u^2 - 1)^2$. Specific assumptions on the initial data u_0 will be given later in the course of the paper. Here and throughout, Δ denotes the Laplacian. When $\gamma = 0$ in (1.1), we obtain the Fisher–Kolmogorov equation that occurs in the study of front propagation [1,2] into unstable states. The model problem (1.1)–(1.3) is proposed in [3,4] as an extension of the Fisher–Kolmogorov equation for the study of spatial patterns in bistable systems. When γ is small ($\gamma \leq 1/8$), it is observed in [4,5] that the solutions of EFK equations are similar to the FK equation but lead to smooth fronts. However when $\gamma > 1/8$, it is possible to distinguish the solutions of these equations [5]. The study of the dynamics of the EFK equation can be found in [6,5,7].

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From the numerical point of view, a conforming finite element method has been proposed in [8] for the EFK equation (1.1) and the error analysis has been discussed. In this article, we propose and study the C^0 interior penalty method for (1.1). In the past few years, C^0 interior penalty methods [9,10] have became an attractive alternative for fourth order problems since the design of quasi-optimal C^0 interior penalty methods is straightforward [11,12]. During the past few years, fully discontinuous Galerkin methods have also became attractive for fourth order problems [13–17] although they involve a larger number of degrees of freedom than the C^0 interior penalty method. C^0 interior penalty methods are designed based on a standard Lagrange finite element space and a mesh dependent weak formulation involving jumps of the normal derivative across the inter-element boundaries. Since the standard Lagrange finite element spaces are designed for second order problems, they are naturally suitable for singularly perturbed fourth order problems. In [18], a C^0 interior penalty method is analyzed for a singularly perturbed fourth order elliptic problem and proved to be robust with respect to the small perturbation parameter. In this article, we extend the results in [18] to study a fully discrete C^0 interior penalty method for the EFK equation (1.1) involving a singularly perturbed fourth order term (when γ is small). We establish the stability of the numerical solution and derive *a priori* error estimates which are robust in γ (depend on a lower order polynomial in γ^{-1}). To accomplish this, we establish a regularity estimate for the solution u of (1.1) that is explicit in γ and derive stability bounds for an elliptic projection of u.

The rest of the article is organized as follows. In Section 2, we derive *a priori* bounds for the solution of the EFK equation. Therein, we propose our numerical method and prove the existence and uniqueness of a discrete solution. Moreover, we derive stability estimates for both the weak and the discrete solution. In Section 3, we derive some regularity of the weak solution. In Section 4, *a priori* error estimates that are robust in γ are derived. Finally, we present conclusions in Section 5.

2. Existence and uniqueness results

In this section, we present a fully discrete C^0 interior penalty method and show the existence and uniqueness of the discrete solution. We derive some *a priori* bounds for the solution of the EFK equation and its discrete counterpart.

Let $V = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega\}$. Denote the $L_2(\Omega)$ inner-product by $(\cdot,\ \cdot)$ and the norm by $\|\cdot\|$.

The weak form of (1.1)–(1.3) is to find $u(\cdot, t) \in V$, $t \in [0, T]$ such that

$$(u_t, v) + \gamma(\Delta u, \Delta v) + (\nabla u, \nabla v) + (\phi(u), v) = 0, \quad \forall v \in V, \tag{2.1}$$

$$u = u_0 \quad \text{at } t = 0. \tag{2.2}$$

The following lemma on the norm equivalence is useful in our analysis.

Lemma 2.1. *There exist two positive constants C_1 and C_2 which may depend on Ω such that*

$$C_1\|v\|_{H^2(\Omega)} \le \left(\|\Delta v\|^2 + \|v\|^2_{H^1(\Omega)}\right)^{1/2} \le C_2\|v\|_{H^2(\Omega)}, \quad \forall v \in V. \tag{2.3}$$

Proof. Let $v \in V$. Then it is obvious that $\left(\|\Delta v\|^2 + \|v\|^2_{H^1(\Omega)}\right)^{1/2} \le C_2\|v\|_{H^2(\Omega)}$.

To prove the other way, note that $v \in V$ satisfies the following elliptic problem:

$$-\Delta v = -\Delta v \quad \text{in } \Omega,$$
$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Appealing to the elliptic regularity theory for the second order homogeneous Neumann problem on convex polyhedral domains [19,20], there exists some positive constant C depending on Ω such that

$$\|\dot{v}\|_{H^2(\Omega)/R} \le C\left(\|\Delta v\| + \|\partial v/\partial n\|_{H^{1/2}(\partial\Omega)}\right) = C\|\Delta v\|, \tag{2.4}$$

where $\dot{v}$ denotes the equivalence class of v in the quotient space $H^2(\Omega)/R$ equipped with the norm

$$\|\dot{v}\|_{H^2(\Omega)/R} = \inf_{c\in R}\|v + c\|_{H^2(\Omega)} = \inf_{c\in R}\left(\|v + c\|^2_{L_2(\Omega)} + \|\nabla v\|^2_{L_2(\Omega)} + |v|^2_{H^2(\Omega)}\right)^{1/2}. \tag{2.5}$$

From (2.4) and (2.5), we find

$$|v|_{H^2(\Omega)} \le C\|\Delta v\|.$$

This completes the proof. $\square$

We derive some *a priori* bounds for u which are explicit in terms of the parameter γ. Throughout the article, C and $C(T)$ denote generic positive constants which are independent of the constant γ.