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Real zeros of ${}_2F_1$ hypergeometric polynomials

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ABSTRACT

We use a method based on the division algorithm to determine all the values of the real parameters b and c for which the hypergeometric polynomials ${}_2F_1(-n, b; c; z)$ have n real, simple zeros. Furthermore, we use the quasi-orthogonality of Jacobi polynomials to determine the intervals on the real line where the zeros are located.

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1. Introduction

The ${}_2F_1$ hypergeometric function is defined by (cf. [1])

$$ {}_2F_1\,(a, b; c; z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1, $$

where a, b and c are complex parameters, $-c \notin \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and

$$ (\alpha)_k = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+k-1), & k \in \mathbb{N}, \\ 1, & k = 0, \alpha \neq 0 \end{cases} $$

is Pochhammer's symbol. This series converges when $|z| < 1$ and also when $z = 1$ provided that $\mathrm{Re}(c - a - b) > 0$ and when $z = -1$ provided that $\mathrm{Re}(c - a - b + 1) > 0$. When one of the numerator parameters is equal to a nonpositive integer, say $a = -n$, $n \in \mathbb{N}_0$, the series terminates and the function is a polynomial of degree n in z.

The problem of describing the zeros of the polynomials ${}_2F_1\,(-n, b; c; z)$ when b and c are complex arbitrary parameters, has not been solved. Even when b and c are both real, the only cases that have been fully analysed impose additional restrictions on b and c. Recent publications (cf. [2–7]) considered the zero location of special classes of ${}_2F_1\,(-n, b; c; z)$ with restrictions on the parameters b and c. Results on the asymptotic zero distribution of certain classes of ${}_2F_1\,(-n, b; c; z)$ have also appeared (cf. [8–12]).

Different types of ${}_2F_1\,(-n, b; c; z)$ have well-established connections with classical orthogonal polynomials, notably the Jacobi polynomials and the Gegenbauer or ultraspherical polynomials (cf. [1]). For the ranges of the parameters where these

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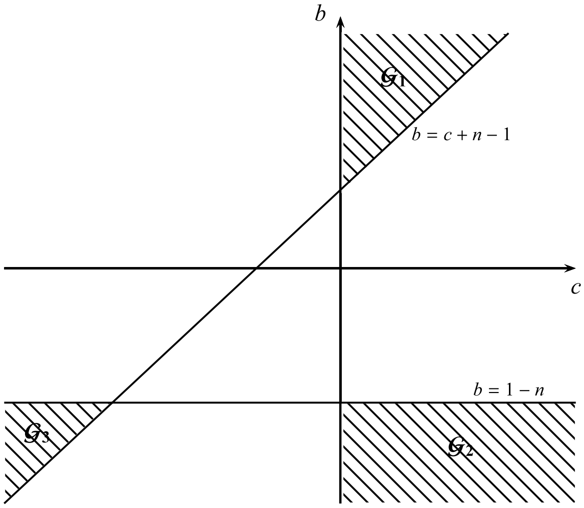

Fig. 1. Values of b and c for which ${}_2F_1(-n, b; c; z)$ is orthogonal and has n real simple zeros in the intervals $(0, 1)$, $(-\infty, 0)$ and $(1, \infty)$ are indicated by regions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ respectively.

polynomials are orthogonal, information about the zeros of ${}_2F_1(-n, b; c; z)$ follows immediately from classical results (cf. [1,13]). The asymptotic zero distribution of ${}_2F_1(-n, b; c; z)$ when b and c depend on n can be deduced from recent results by Kuijlaars, Martínez-Finkelshtein, Martínez-González and Orive (cf. [14–17]) on the asymptotic zero distribution of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ when the parameters α and β depend on n. Conversely, if the distribution of the zeros of ${}_2F_1(-n, b; c; z)$ is known, this leads to information about the zero distribution of other special functions (cf. [3]). This makes knowledge of the zero distribution of ${}_2F_1(-n, b; c; z)$ extremely valuable.

The orthogonality of the polynomials ${}_2F_1(-n, b; c; z)$ given in the next theorem follows from the orthogonality of the Jacobi polynomials (cf. [18, pp. 257–261]) and can also be proved directly using the Rodrigues' formula for the polynomials ${}_2F_1(-n, b; c; z)$ (cf. [1, p. 99]) as was done in [19,15].

Theorem 1 *(cf. [19]). Let $n \in \mathbb{N}_0$, $b, c \in \mathbb{R}$ and $-c \notin \mathbb{N}_0$. Then ${}_2F_1(-n, b; c; z)$ is the nth degree orthogonal polynomial for the n-dependent positive weight function $|z^{c-1}(1-z)^{b-c-n}|$ on the intervals*

(i) $(-\infty, 0)$ *for* $c > 0$ *and* $b < 1 - n$;
(ii) $(0, 1)$ *for* $c > 0$ *and* $b > c + n - 1$;
(iii) $(1, \infty)$ *for* $c + n - 1 < b < 1 - n$.

As a consequence of orthogonality, we know that for each n, the n zeros of ${}_2F_1(-n, b; c; z)$ are real, simple and lie in the interval of orthogonality for the corresponding ranges of the parameters (see, for example, [20, Theorem 4]) as illustrated in Fig. 1.

In his classical paper (cf. [21]), Felix Klein obtained results on the precise number of zeros of ${}_2F_1(a, b; c; z)$ that lie in each of the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, \infty)$ by generalizing earlier results of Hilbert (cf. [22]). These Hilbert–Klein formulae are valid for hypergeometric functions and not only for polynomials. Szegö recaptured these results for the special case of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, which have a representation as ${}_2F_1(-n, b; c; z)$, in the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$ (cf. [13, p. 145, Theorem 6.72]). The number and location of the real zeros of ${}_2F_1(-n, b; c; z)$ for b and c real can be deduced as follows.

Theorem 2 *(cf. [6, Theorem 3.2]). Let $n \in \mathbb{N}$, $b, c \in \mathbb{R}$ and $c > 0$. Then,*

(i) *For $b > c + n$, all zeros of ${}_2F_1(-n, b; c; z)$ are real and lie in the interval $(0, 1)$.*
(ii) *For $c < b < c + n$, $c + j - 1 < b < c + j$, $j = 1, 2, \ldots, n$, ${}_2F_1(-n, b; c; z)$ has j real zeros in $(0, 1)$. The remaining $(n - j)$ zeros of ${}_2F_1(-n, b; c; z)$ are all non-real if $(n - j)$ is even, while if $(n - j)$ is odd, ${}_2F_1(-n, b; c; z)$ has $(n - j - 1)$ non-real zeros and one additional real zero in $(1, \infty)$.*
(iii) *For $0 < b < c$, all the zeros of ${}_2F_1(-n, b; c; z)$ are non-real if n is even, while if n is odd, ${}_2F_1(-n, b; c; z)$ has one real zero in $(1, \infty)$ and the other $(n - 1)$ zeros are non-real.*
(iv) *For $-n < b < 0$, $-j < b < -j + 1$, $j = 1, 2, \ldots, n$, ${}_2F_1(-n, b; c; z)$ has j real negative zeros. The remaining $(n - j)$ zeros of ${}_2F_1(-n, b; c; z)$ are all non-real if $(n - j)$ is even, while if $(n - j)$ is odd, ${}_2F_1(-n, b; c; z)$ has $(n - j - 1)$ non-real zeros and one additional real zero in $(1, \infty)$.*
(v) *For $b < -n$, all zeros of ${}_2F_1(-n, b; c; z)$ are real and negative.*

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