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A comparison study of ADI and operator splitting methods on option pricing models

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ABSTRACT

In this paper we perform a comparison study of alternating direction implicit (ADI) and operator splitting (OS) methods on multi-dimensional Black–Scholes option pricing models. The ADI method is used extensively in mathematical finance for numerically solving multi-factor option pricing problems. However, numerical results from the ADI scheme show oscillatory solution behaviors with nonsmooth payoffs or discontinuous derivatives at the exercise price with large time steps. In the ADI scheme, there are source terms which include y-derivatives when we solve x-derivative involving equations. Then, due to the nonsmooth payoffs, source terms contain abrupt changes which are not in the range of implicit discrete operators and this leads to difficulty in solving the problem. On the other hand, the OS method does not contain the other variable's derivatives in the source terms. We provide computational results showing the performance of the methods for two-asset option pricing problems. The results show that the OS method is very efficient and gives better accuracy and robustness than the ADI method with large time steps.

1. Introduction

In today's financial markets, options are the most common securities that are frequently bought and sold. Under the Black–Scholes partial differential equation (BS PDE) framework, various numerical methods (see e.g., [1–5]) have been presented by using the finite difference method (FDM) to solve the option pricing problems (see e.g., [6–12]). However most option pricing problems have nonsmooth payoffs or discontinuous derivatives at the exercise price. Standard finite difference schemes used to solve the problems with nonsmooth payoffs and large time steps do not work well because of discontinuities introduced in the source terms. Moreover, these unwanted oscillations become problematic when we estimate the Greeks, the hedging parameters such as Delta, Gamma, Rho, Theta, and Vega.

Let $s_i(t),\ i = 1, 2, \dots, d$ denote the value of the underlying i-th asset at time t and $u(\mathbf{s}, t)$ denote the price of an option. Here, $\mathbf{s} = (s_1, s_2, \dots, s_d)$. In the Black–Scholes model [13], each underlying asset $s_i(t)$ satisfies the following stochastic differential equation:

$$ds_i(t) = \mu_i s_i(t)dt + \sigma_i s_i(t)dW_i(t), \quad i = 1, 2, \dots, d,$$

where μ_i, σ_i, and $W_i(t)$ are the expected instantaneous rate of return, constant volatility, and standard Brownian motion on the underlying asset s_i, respectively. And the term dW contains the randomness which is certainly a feature of asset prices and is assumed to be a Wiener process. The Wiener processes are correlated by $\langle dW_i dW_j \rangle = \rho_{ij} dt$. Then the generalized BS

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PDE can be derived by using Ito's lemma and the no-arbitrage principle:

$$\frac{\partial u(\mathbf{s},t)}{\partial t} + \sum_{i=1}^{d} r s_i \frac{\partial u(\mathbf{s},t)}{\partial s_i} + \frac{1}{2}\sum_{i,j=1}^{d} \rho_{ij}\sigma_i\sigma_j s_i s_j \frac{\partial^2 u(\mathbf{s},t)}{\partial s_i \partial s_j} - r u(\mathbf{s},t) = 0,$$
$$u(\mathbf{s},T) = \Lambda(\mathbf{s}),$$

where $r > 0$ is a constant riskless interest rate and $\Lambda(\mathbf{s})$ is the payoff function.

This paper is organized as follows. In Section 2, we introduce the Black–Scholes model in two-dimensional space and describe the ADI and OS numerical methods for the BS PDE. In Section 3, we present several numerical results showing the performance of the standard ADI and OS methods. Then we summarize our results in Section 4.

2. ADI and OS methods for the BS equation

In this paper, we focus on the two-dimensional Black–Scholes equation. Let $\mathcal{L}_{\mathrm{BS}}$ be the operator

$$\mathcal{L}_{\mathrm{BS}} = \frac{1}{2}\sigma_1^2 x^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\sigma_2^2 y^2 \frac{\partial^2 u}{\partial y^2} + \rho\sigma_1\sigma_2 xy \frac{\partial^2 u}{\partial x \partial y} + rx\frac{\partial u}{\partial x} + ry\frac{\partial u}{\partial y} - ru.$$

Then the Black–Scholes equation can be written as

$$\frac{\partial u}{\partial \tau} = \mathcal{L}_{\mathrm{BS}} \quad \text{for } (x,y,\tau) \in \Omega \times (0,T], \tag{1}$$

where $\tau = T - t$. Originally, the option pricing problems are defined in the unbounded domain $\Omega \times (0,T] = \{(x,y,t) \mid x > 0,\ y > 0,\ \tau \in (0,T]\}$. However, we need to truncate this unbounded domain into a finite computational domain in order to solve Eq. (1) numerically by a finite difference method. Therefore, we consider Eq. (1) on a finite domain: $(0,L)\times(0,M)\times(0,T]$, where L and M are large enough so that the error in the price u is negligible. Let us first discretize the given computational domain $\Omega = (0,L)\times(0,M)$ with a uniform space step $h = L/N_x = M/N_y$ and a time step $\Delta\tau = T/N_\tau$. Here, N_x, N_y, and N_τ are the number of grid points in the x-, y-, and τ-direction, respectively. Furthermore, let us denote the numerical approximation of the solution by $u_{ij}^n \equiv u(x_i, y_j, \tau^n) = u(ih, jh, n\Delta\tau)$, where $i = 0,\ldots,N_x, j = 0,\ldots,N_y$, and $n = 0,\ldots,N_\tau$. We use the vertex-centered discretization since we will use a linear boundary condition [7,14–16]: $\frac{\partial^2 u}{\partial x^2}(0,y,\tau) = \frac{\partial^2 u}{\partial x^2}(L,y,\tau) = \frac{\partial^2 u}{\partial y^2}(x,0,\tau) = \frac{\partial^2 u}{\partial y^2}(x,M,\tau) = 0$, for $0 \le x \le L,\ 0 \le y \le M,\ 0 \le \tau \le T$.

2.1. Alternating directions implicit method

The main idea of the ADI method (see e.g., [17,18]) is to proceed in two stages, treating only one operator implicitly at each stage. First, a half-step is taken implicitly in x and explicitly in y. Then, the other half-step is taken implicitly in y and explicitly in x. The full scheme is

$$\frac{u_{ij}^{n+\frac{1}{2}} - u_{ij}^n}{\Delta\tau} = \mathcal{L}_{\mathrm{ADI}}^x u_{ij}^{n+\frac{1}{2}}, \tag{2}$$

$$\frac{u_{ij}^{n+1} - u_{ij}^{n+\frac{1}{2}}}{\Delta\tau} = \mathcal{L}_{\mathrm{ADI}}^y u_{ij}^{n+1}, \tag{3}$$

where the discrete difference operators $\mathcal{L}_{\mathrm{ADI}}^x$ and $\mathcal{L}_{\mathrm{ADI}}^y$ are defined by

$$\begin{aligned}
\mathcal{L}_{\mathrm{ADI}}^x u_{ij}^{n+\frac{1}{2}} = {} & \frac{(\sigma_1 x_i)^2}{4}\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{ij}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{h^2} + \frac{(\sigma_2 y_j)^2}{4}\frac{u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n}{h^2} \\
& + \frac{1}{2}\rho\sigma_1\sigma_2 x_i y_j \frac{u_{i+1,j+1}^n + u_{i-1,j-1}^n - u_{i-1,j+1}^n - u_{i+1,j-1}^n}{4h^2} \\
& + \frac{1}{2} r x_i \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i,j}^{n+\frac{1}{2}}}{h} + \frac{1}{2} r y_j \frac{u_{ij+1}^n - u_{ij}^n}{h} - \frac{1}{2} r u_{ij}^{n+\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{ADI}}^y u_{ij}^{n+1} = {} & \frac{(\sigma_1 x_i)^2}{4}\frac{u_{i+1,j}^{n+\frac{1}{2}} - 2u_{ij}^{n+\frac{1}{2}} + u_{i-1,j}^{n+\frac{1}{2}}}{h^2} + \frac{(\sigma_2 y_j)^2}{4}\frac{u_{i,j+1}^{n+1} - 2u_{ij}^{n+1} + u_{i,j-1}^{n+1}}{h^2} \\
& + \frac{1}{2}\rho\sigma_1\sigma_2 x_i y_j \frac{u_{i+1,j+1}^{n+\frac{1}{2}} + u_{i-1,j-1}^{n+\frac{1}{2}} - u_{i-1,j+1}^{n+\frac{1}{2}} - u_{i+1,j-1}^{n+\frac{1}{2}}}{4h^2} \\
& + \frac{1}{2} r x_i \frac{u_{i+1,j}^{n+\frac{1}{2}} - u_{i,j}^{n+\frac{1}{2}}}{h} + \frac{1}{2} r y_j \frac{u_{ij+1}^{n+1} - u_{ij}^{n+1}}{h} - \frac{1}{2} r u_{ij}^{n+1}.
\end{aligned}$$