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On Hermite interpolation with polynomial splines on T-meshes

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Dedicated to Leonardo Traversoni on the occasion of his 60th birthday

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ABSTRACT

Hermite interpolation using polynomial splines on T-meshes is discussed in detail, leading to an error bound for interpolation of smooth functions.

1. Introduction

This paper is a continuation of our recent paper [1] in which we established the approximation power of polynomial splines defined on T-meshes. The proof was based on the construction of a stable local basis and an associated quasi-interpolation operator. In this paper we provide an alternative approach to getting the approximation power of the same spline spaces. It is based on an explicit Hermite interpolation operator which may be of interest in its own right. Polynomial splines on T-meshes have a variety of applications in data fitting, surface modeling, and the numerical solution of boundary value problems, see e.g. [2–5].

A T-mesh $\triangle := \{R_i\}_{i=1}^N$ is a collection of axis-aligned rectangles such that the interior of the domain $\Omega := \cup R_i$ is connected, and any pair of distinct rectangles can intersect each other only at points on their edges. The domain Ω need not be rectangular, and may have one or more holes, see Fig. 1. T-meshes include tensor-product meshes as a special case. However, in contrast to tensor-product meshes, T-meshes are allowed to have T-nodes, where a T-node is a vertex of one rectangle that lies in the interior of an edge of another rectangle. We also refer to these as hanging vertices, cf. [6,7]. Given $\mathbf{d} := (d_1, d_2)$ and $\mathbf{r} = (r_1, r_2)$, the spline space of interest is

$$\mathcal{S}_{\mathbf{d}}^{\mathbf{r}}(\triangle) := \{s \in C^{\mathbf{r}}(\Omega) : s|_{R_i} \in \mathcal{P}_{\mathbf{d}} \text{ for all } i = 1, \ldots, N\},$$

where

$$\mathcal{P}_{\mathbf{d}} := \text{span}\{x^i y^j\}_{i=0,j=0}^{d_1,d_2}$$

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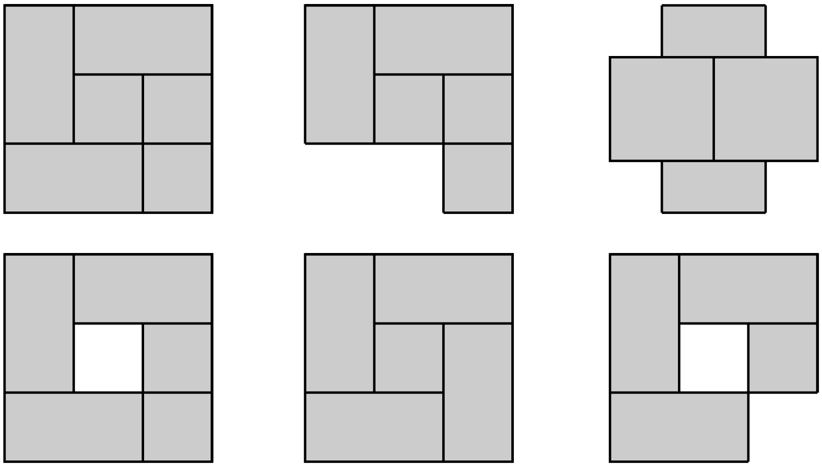

Fig. 1. Some T-meshes. The first four are regular and cycle-free. The fifth one is regular but has a cycle, and the last one is not regular.

is the usual space of tensor product polynomials of degree d_1 in x and degree d_2 in y. Here $C^{\mathbf{r}}(\Omega)$ denotes the space of functions s such that their mixed derivatives $D_x^i D_y^j s$ are continuous for all $0 \le i \le r_1$ and $0 \le j \le r_2$.

It was recognized already in [2,3] that polynomial splines on T-meshes can be used to solve certain Hermite interpolation problems, see Remark 1. Our aim here is to discuss Hermite interpolation with splines on T-meshes in the case $d_1 \ge 2r_1 + 1$ and $d_2 \ge 2r_2 + 1$ in more generality and in more detail, and to derive error bounds for how well smooth functions are approximated by such interpolants.

Throughout the paper we make use of the notation and terminology employed in [1]. For convenience, we recall the main concepts. A T-mesh is called regular provided that for every vertex v, the union of the rectangles containing v has a connected interior. A composite edge of a T-mesh $\triangle$ is a line segment e connecting two vertices of $\triangle$ such that all vertices lying in the interior of e are hanging vertices, and e cannot be extended to a longer line segment with the same property. A cycle is a collection of hanging vertices $w_1, \dots, w_n$ such that for each $i = 1, \dots, n$, the vertex w_i lies in the interior of a composite edge with one endpoint at w_{i+1}, where we set $w_{n+1} = w_1$. A chain ending on a composite edge e is a maximal sequence of composite edges $e_1, \dots, e_m$ such that for each $i = 1, \dots, m$, one end of e_i is in the interior of e_{i+1}, where $e_{m+1} = e$. We call m the length of the chain. In this paper, we focus on T-meshes that are regular and have no cycles. The first four T-meshes in Fig. 1 are regular and cycle-free. The fifth mesh is regular, but has a cycle, while the last one is not regular.

The paper is organized as follows. In Section 2 we discuss Hermite interpolation with $\mathcal{S}_{\mathbf{d}}^{\mathbf{r}}(\triangle)$ and give an explicit constructive algorithm for carrying it out. In Section 3 we show that the Hermite interpolation problem leads immediately to a nodal minimal determining set for $\mathcal{S}_{\mathbf{d}}^{\mathbf{r}}(\triangle)$ which can be used to recover a known dimension formula for $\mathcal{S}_{\mathbf{d}}^{\mathbf{r}}(\triangle)$. The main result of the paper is contained in Section 4, where we give an error bound for how well the Hermite interpolating splines approximate smooth functions and their derivatives. In Sections 5 and 6 we provide some details for the univariate and tensor product Hermite interpolation operators needed to prove our main approximation result. Section 7 includes a bound on the spline interpolation operator. In Section 8 we derive an error bound for rather general tensor product interpolation methods. We conclude the paper with remarks and references.

2. A Hermite interpolating spline

Suppose e is either a horizontal composite edge or a horizontal edge of a rectangle of $\triangle$. Then we define $\Lambda_e := \{\xi_{e,i}\}_{i=1}^{d_1-2r_1-1}$ to be the set of equally spaced points in the interior of e. Similarly, for every vertical composite edge e or a vertical edge of a rectangle, let $\Lambda_e := \{\xi_{e,j}\}_{j=1}^{d_2-2r_2-1}$ be the set of equally spaced points in the interior of e. For every rectangle R in $\triangle$, let

$$\Lambda_R := \{u_{ij}^R\} := \{(\xi_{e,i}^x, \xi_{\tilde{e},j}^y)\}_{i,j=1}^{d_1-2r_1-1, d_2-2r_2-1}, \tag{2.1}$$

where e and $\tilde{e}$ are the bottom and left edges of R, and the superscripts x and y denote the x and y coordinates, respectively. Our first result describes Hermite interpolation with $\mathcal{S}_{\mathbf{d}}^{\mathbf{r}}(\triangle)$.

Theorem 2.1. *Suppose* $\mathbf{d} \ge 2\mathbf{r} + 1$, *i.e.,* $d_1 \ge 2r_1 + 1$ *and* $d_2 \ge 2r_2 + 1$. *Then the values*

(1) $\{D_x^i D_y^j s(v)\}_{i=0,j=0}^{r_1,r_2}$ *for every nonhanging vertex* v,

(2) $\{D_y^j s(\xi_{e,i})\}_{i=1,j=0}^{d_1-2r_1-1,r_2}$ *for every horizontal composite edge* e,