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Split Bregman iteration and infinity Laplacian for image decomposition

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ABSTRACT

In this paper, we address the issue of decomposing a given real-textured image into a cartoon/geometric part and an oscillatory/texture part. The cartoon component is modeled by a function of bounded variation, while, motivated by the works of Meyer [Y. Meyer, Oscillating Patterns in Image Processing and Nonlinear Evolution Equations, vol. 22 of University Lecture Series, AMS, 2001], we propose to model the oscillating component v by a function of the space G of oscillating functions, which is, in some sense, the dual space of $BV(\Omega)$. To overcome the issue related to the definition of the G-norm, we introduce auxiliary variables that naturally emerge from the Helmholtz–Hodge decomposition for smooth fields, which yields to the minimization of the L^∞-norm of the gradients of the new unknowns. This constrained minimization problem is transformed into a series of unconstrained problems by means of Bregman Iteration. We prove the existence of minimizers for the involved subproblems. Then a gradient descent method is selected to solve each subproblem, becoming related, in the case of the auxiliary functions, to the infinity Laplacian. Existence/Uniqueness as well as regularity results of the viscosity solutions of the PDE introduced are proved.

1. Introduction and related prior works

We limit the presentation to the two-dimensional case and to grey-scale images although the method can be extended to higher dimensions and to vector-valued data. Let Ω be an open, bounded and connected domain in $\mathbb{R}^2$ with Lipschitz continuous boundary and let $f : \Omega \to \mathbb{R}$ be a given observed image function. Decomposition techniques consist in separating the geometric component u of f from the oscillatory part v. More precisely, as stressed in [1], the decomposition of f into $u + v$ can be phrased as a functional minimization problem of the following kind:

$$\inf_{(u,v)\in X_1\times X_2} \{F_1(u) + \lambda F_2(v),\ f = u + v\},$$

with $F_1,\ F_2 \geq 0$ two functionals and $\lambda > 0$, a tuning parameter. In order for this problem to be well-posed, it is required that $X_1 = \{u,\ F_1(u) < \infty\}$ and $X_2 = \{v,\ F_2(v) < \infty\}$, and $f \in X_1 + X_2$. Also, $F_1(u)$ and $F_2(v)$ must be small, and $F_1(v) > F_1(u)$, $F_2(u) > F_2(v)$ insuring that the two components can be properly discriminated. The choice $X_1 = BV(\Omega)$ is well-suited when representing homogeneous regions with sharp edges. In [2], Meyer shows that if the residual v defined by $v = f - u$ represents oscillations/texture or noise, a suitable space is the Banach space of generalized functions $v(x, y)$ which can be written as

$$v(x, y) = \partial_x g_1(x, y) + \partial_y g_2(x, y),$$

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$g_1, g_2 \in L^\infty(\mathbb{R}^2)$, induced by the norm $\|v\|_*$ defined by:

$$\|v\|_* = \inf_{\vec{g}=(g_1,g_2)\in(L^\infty(\mathbb{R}^2))^2,\ v=\operatorname{div}\vec{g}} \|\,|\vec{g}|\,\|_{L^\infty(\mathbb{R}^2)},$$

with $|\vec{g}(x,y)| = \sqrt{g_1^2(x,y)+g_2^2(x,y)}$. The texture component is, in particular, better modeled than with the L^2-space. Also, alternative spaces could be considered such as the space F defined as G but with g_1 and g_2 belonging to the John and Nirenberg space $BMO(\Omega)$ (see [1] or [2] for further details). The author then proposes the following image decomposition model:

$$\inf_u \int_\Omega |\nabla u| + \lambda \|f-u\|_*,$$

with $\int_\Omega |\nabla u|$ the total variation of u.

It is clear that this convex minimization problem cannot be directly solved in practice, owing to the particular form of the $\|\cdot\|_*$ norm. Some prior related works focused on approximations of this model. Motivated by the approximation of the L^∞-norm of $|\vec{g}|$:

$$\left\|\sqrt{g_1^2+g_2^2}\right\|_{L^\infty} = \lim_{p\to+\infty} \left\|\sqrt{g_1^2+g_2^2}\right\|_{L^p}.$$

Vese and Osher propose in [3] the following convex minimization problem:

$$\inf_{u,g_1,g_2}\left\{G_p(u,g_1,g_2) = \int_\Omega |\nabla u| + \lambda\int_\Omega |f-u-\operatorname{div}\vec{g}|^2\,dx + \mu\left[\int_\Omega\left(\sqrt{g_1^2+g_2^2}\right)^p dx\right]^{\frac{1}{p}}\right\}.$$

The first term guarantees that $u \in BV(\Omega)$, while the second ensures that $\operatorname{div}\vec{g}$ is close to $f-u$ and the last term penalizes the L^p-norm of $|\vec{g}|$. Thus formally when $\lambda \to +\infty$ and $p \to +\infty$, the model is an approximation of the (BV, G) model by Meyer.

The case $p = 2$ corresponds to the space $H^{-1}(\Omega)$ and is addressed in [4]. The authors assume the existence of a unique Hodge decomposition of $\vec{g}$ as $\vec{g} = \nabla P + \vec{Q}$ with $\vec{Q}$ a divergence free vector field that is neglected afterwards. Consequently, $v = f-u = \operatorname{div}\vec{g} = \Delta P$. It can be proved that for each $v \in L^2(\Omega)$ with $\int_\Omega v(x,y)\,dx\,dy = 0$, there is a unique $P \in H^1(\Omega)$ such that $v = -\Delta P$, $\int_\Omega P(x)\,dx = 0$ and $\frac{\partial P}{\partial \vec{n}} = 0$ on $\partial\Omega$. This is then expressed by $P = \Delta^{-1}v = \Delta^{-1}(f-u)$ and the introduced minimization problem is phrased using the H^{-1}-norm $\|v\|^2_{H^{-1}(\Omega)} = \int_\Omega |\nabla(\Delta^{-1})(v)|^2\,dx$.

Aujol et al. [5] propose another approximation of the (BV, G) model by minimizing:

$$\inf_{(u,v)\in BV(\Omega)\times G_\mu(\Omega)} \int_\Omega |\nabla u| + \frac{1}{2\lambda}\|f-u-v\|^2_{L^2(\Omega)},$$

with $v \in L^2(\Omega)\cap G(\Omega)$ and where $G_\mu(\Omega) = \{v \in G,\ \|v\|_* \le \mu\}$.

Recently, Elion and Vese [1] have presented a model in which the cartoon part is modeled by a function of bounded variation and the oscillatory part as the Laplacian of a single-valued function whose gradient belongs to L^∞. This is again motivated by the decomposition of the field $\vec{g}$ into $\nabla P + \vec{Q}$, $\vec{Q}$ being a divergence-free vector field that is neglected afterwards. Given $f \in L^2(\Omega)$, the proposed model is:

$$\inf_{\substack{u\in BV(\Omega),\ \nabla P\in(L^\infty(\Omega))^2\\ \Delta P\in L^2(\Omega)}} \int_\Omega |\nabla u| + \mu\int_\Omega |f-(u+\Delta P)|^2\,dx + \lambda\|\nabla P\|_{L^\infty(\Omega)}$$

and is related to the absolutely minimizing Lipschitz extensions. A major difference with our proposed model is that we use a split Bregman iteration approach, which avoids getting a fourth-order term in the Euler–Lagrange equations that is difficult to handle numerically. Also, we consider the general Helmholtz–Hodge decomposition of smooth 2D vector fields without neglecting the divergence-free component. At last, several theoretical results are provided.

To conclude this part and for the sake of completeness, we refer the reader to [6–9] for other outlooks of the problem of image decomposition.

2. Modeling and theoretical results

2.1. Modeling

As previously stressed, the (BV, G) model of Meyer cannot be directly solved in practice, due to the particular form of the $\|\cdot\|_*$-norm. We thus propose to decompose the vector field $\vec{g}$ by means of the Helmholtz–Hodge theorem (see [10] or [11]). For smooth data, the Helmholtz–Hodge decomposition of 2D vectors $\vec{g}$ can be formulated as follows:

$$\begin{aligned}\vec{g} &= \nabla d + J\nabla r + \vec{h},\\ &= D + R + \vec{h},\end{aligned}$$