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A multiplier version of the Bernstein inequality on the complex sphere

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ABSTRACT

We prove a multiplier version of the Bernstein inequality on the complex sphere. Included in this is a new result relating a bivariate sum involving Jacobi polynomials and Gegenbauer polynomials, which relates the sum of reproducing kernels on spaces of polynomials irreducibly invariant under the unitary group, with the reproducing kernel of the sum of these spaces, which is irreducibly invariant under the action of the unitary group.

1. Introduction and preliminaries

In this article we prove a multiplier version of the Bernstein inequality of the type proved by Ditzian [1]. He considers an operator Λ defined by a sequence of multipliers $\lambda_1, \lambda_2, \ldots,$ on a set of polynomial subspaces $H_1, H_2, \ldots,$ with $P_n = \bigcup_{i=1}^n H_n$, and the completion B of $\bigcup_{n=1}^\infty H_n$ in some norm $\|\cdot\|$. If the reproducing kernels for P_n have norm bounded kth order Cesàro means for some k then he shows that the multiplier-type Bernstein inequality

$$\|\Lambda p_n\| \le C\lambda_n \|p_n\|, \quad p_n \in P_n, \tag{1}$$

holds. Particular sequences of multipliers realise the Laplace operator on compact Reimannian manifolds, but it is only for a narrow class of spaces that the boundedness of the Cesàro means of the reproducing kernels is known. These include the two-point homogeneous spaces, including the spheres, and projective spaces, where the reproducing kernels are essentially univariate orthogonal polynomials. This is not the case for the complex spheres, where the structure of the reproducing kernels is more complicated.

We focus on perturbations of the Laplace operator on the complex sphere and show the same result as (1) above in Theorem 4.1. This is not a direct consequence of the results on the real sphere, because the pseudodifferential operators are defined on a finer invariant division of the harmonic subspaces than is present on the real sphere.

The classical Bernstein inequality is related to classical derivatives on the manifold in question. This paper does not address this subject, which has a rich history of its own. We just remark that, since the restriction to a geodesic of a polynomial on a complex sphere is just a trigonometric polynomial on a circle, we immediately have a *tangential Bernstein inequality*

$$\|D_u p_n\|_\infty \le n\|p_n\|_\infty,$$

where D_u is the tangential derivative in the direction of u and p_n is any polynomial of degree n. For more information on tangential Bernstein inequalities on algebraic manifolds see e.g. Bos et al. [2].

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An important consequence of the development of ideas in this paper is Theorem 2.1, in which is proved a new bivariate summation formula for Jacobi polynomials.

We follow Koornwinder [3] in our description of the complex sphere, and the harmonic analysis thereof. Let $\mathbb{C}^q$ be a q-dimensional complex space. We will denote vectors in $\mathbb{C}^q$ by $\mathbf{z} = (z_1, z_2, \ldots, z_q)$. Let the inner product of two vectors $\mathbf{w}, \mathbf{z} \in \mathbb{C}^q$ be

$$\langle \mathbf{w}, \mathbf{z}\rangle = \sum_{j=1}^{q} w_j \bar{z}_j,$$

and the length of a vector be $|\mathbf{z}| = \langle \mathbf{z}, \mathbf{z}\rangle^{1/2}$. Let

$$\mathbb{S}^{2q} = \{\mathbf{z} \in \mathbb{C}^q : |\mathbf{z}| = 1\},$$

be the sphere in $\mathbb{C}^q$. We note here that $\mathbb{S}^{2q}$ has topological dimension $2q-1$, but that we keep with the established notation so as not to confuse the reader. Let $d(\mathbf{w}, \mathbf{z})$ be the geodesic distance between $\mathbf{w}$ and $\mathbf{z}$ on $\mathbb{S}^{2q}$.

The complex sphere is invariant under the action of the unitary group $\mathcal{U}_q$, the group of $q \times q$ complex matrices U which satisfy

$$UU^* = I_q,$$

where $U^*_{ij} = \overline{U_{ji}}, i, j = 1, \ldots, q$.

Using the polar form for a complex number we can write $\mathbf{z} \in \mathbb{S}^{2q}$ in the form

$$\mathbf{z} = (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}, \ldots, r_q e^{i\phi_q}),$$

where $\sum_{j=1}^{q} r_j^2 = 1$. If we set $r_1 = \cos\theta$, we can write

$$\mathbf{z} = \cos\theta e^{i\phi}\mathbf{e}_1 + \sin\theta \mathbf{z}', \tag{2}$$

where $\mathbf{e}_k$ is the unit vector in the kth coordinate, and $\mathbf{z}' \in \mathbb{S}^{2(q-1)}$. Here $\phi = \phi_1$, (obviously) $\sin\theta = \sqrt{r_2^2 + \cdots + r_q^2}$, and

$$\mathbf{z}' = (\sin\theta)^{-1}(r_2 e^{i\phi_2}, \ldots, r_q e^{i\phi_q}).$$

We can easily verify that $\mathbb{S}^{2q} = \{U\mathbf{e}_1,\ U \in \mathcal{U}_q\}$. Thus, for any $\mathbf{z} \in \mathbb{S}^{2q}$, there exists a $U \in \mathcal{U}_q$ such that $U\mathbf{e}_1 = \mathbf{z}$. This type of action of $\mathcal{U}_q$ on $\mathbb{S}^{2q}$ is said to be *transitive*. Now it is clear that if we view $\mathcal{U}_{q-1}$ as acting on the orthogonal complement of $\mathbf{e}_1$, then $\mathbf{e}_1$ remains fixed under this action. Thus we can write

$$\mathbb{S}^{2q} = \frac{\mathcal{U}_q}{\mathcal{U}_{q-1}}.$$

On the real sphere we are accustomed to the idea that the polynomials on the sphere may be orthogonally decomposed into subspaces of spherical harmonics, each of which is invariant under the action of the orthogonal group. For the complex sphere the picture is not so straightforward. Now we wish to identify the spaces of polynomials which are minimally invariant under the action of the unitary group, and this issue is discussed in Section 2.

Let $d\mu_{2q}$ be the $\mathcal{U}_q$-invariant normalised measure on the sphere, and define the inner product of f, g, two functions on S^{2q}, by

$$\langle\langle f, g\rangle\rangle = \int_{\mathbb{S}^{2q}} f\bar{g}\, d\mu_{2q}.$$

Let us define the family of L_r norms on $\mathbb{S}^{2q}$:

$$\|f\|_r = \begin{cases} \left(\int_{\mathbb{S}^{2q}} |f|^r d\mu_{2q}\right)^{1/r}, & 1 \le r < \infty, \\ \operatorname{ess\,sup}|f|, & r = \infty. \end{cases}$$

In this paper we will be discussing $\mathcal{U}_q$ invariant kernels on $\mathbb{S}^{2q}$. These are kernels $\kappa : \mathbb{S}^{2q} \times \mathbb{S}^{2q} \to \mathbb{C}$, such that $\kappa(U\mathbf{x}, U\mathbf{y}) = \kappa(\mathbf{x}, \mathbf{y})$ for all $U \in \mathcal{U}_q$. Previous results of Ditzian [1] have been shown to be valid for two-point homogeneous spaces. These are spaces which for pairs of points which are equidistant, there is a single isometry which maps one pair to the other (see Wang [4] for more information). For two points spaces, the geodesic distance is a function of the inner product in the ambient space. A consequence of this is that all isometrically invariant kernels are univariate functions of distance.

For the complex spheres this is not the case. However, we do have the following analogous property. Suppose we have pairs of points $\mathbf{x}_1, \mathbf{y}_1$ and $\mathbf{x}_2, \mathbf{y}_2$, with $\langle \mathbf{x}_1, \mathbf{y}_1\rangle = \langle \mathbf{x}_2, \mathbf{y}_2\rangle$. Since the unitary group acts transitively on the complex sphere, there exist $U_1, U_2 \in \mathcal{U}_q$ such that $U_1\mathbf{x}_1 = U_2\mathbf{x}_2 = \mathbf{e}_1$. Recalling (2), and using the fact that $\mathcal{U}_{q-1}$ acts transitively on $\mathbb{S}^{2q-2}$, we know there exists $U' \in \mathcal{U}_q$, such that $U'U_1\mathbf{y}_1 = U_2\mathbf{y}_2$, and $U'\mathbf{e}_1 = \mathbf{e}_1$. Hence, $U_2^{-1}U'U_1\mathbf{x}_1 = \mathbf{x}_2$, and $U_2^{-1}U'U_1\mathbf{y}_1 = \mathbf{y}_2$. Hence, we conclude that if $\langle \mathbf{x}_1, \mathbf{y}_1\rangle = \langle \mathbf{x}_2, \mathbf{y}_2\rangle$ there exists $U \in \mathcal{U}_q$ such that $U\mathbf{x}_1 = \mathbf{x}_2$ and $U\mathbf{y}_1 = \mathbf{y}_2$. This is analogous to the two point homogeneous property of real spheres. A straightforward consequence of this is that if κ is $\mathcal{U}_q$ invariant $\kappa(\mathbf{x}_1, \mathbf{y}_1) = \kappa(U\mathbf{x}_1, U\mathbf{y}_1) = \kappa(\mathbf{x}_2, \mathbf{y}_2)$, so that κ is invariant on points with $\langle \mathbf{x}_1, \mathbf{y}_1\rangle = \langle \mathbf{x}_2, \mathbf{y}_2\rangle$. Thus we have