



Upscaling for the Laplace problem using a discontinuous Galerkin method

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ABSTRACT

Scientists and engineers generally tackle problems that include multiscale effects and that are thus difficult to solve numerically. The main difficulty is to capture both the fine and the coarse scales to get an accurate numerical solution. Indeed, the computations are generally performed by using numerical schemes based on grids. But the stability and thus the accuracy of the numerical method depends on the size of the grid which must be refined drastically in the case of very fine scales. That implies huge computational costs and in particular the limitations of the memory capacity are often reached. It is thus necessary to use numerical methods that are able to capture the fine scale effects with computations on coarse meshes. Operator-based upscaling is one of them and we present a first attempt to adapt that technique to a Discontinuous Galerkin Method (DGM). We consider the Laplace problem as a benchmark and we compare the performance of the resulting numerical scheme with the classical one using Lagrange finite elements. The comparison involves both an accuracy analysis and a complexity calculus. This work shows that there is an interest of combining DGM with upscaling.

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1. Introduction

The wave propagation is widely used in a large variety of scientific fields as in oil exploration where the issue is to produce images of hydrocarbon stocks that are hidden and nowadays very difficult to detect. The principle is based on the fact that the wave equation can be reversed in time, which means that any arrival time of a reflected wave can be transformed into a spatial measurement providing the localization of the corresponding reflector. From a numerical point of view, the process which is known as the Reverse Time Migration, requires to solve many wave equations in complex media whose tectonic includes strong heterogeneities and the contrasts of the physical parameters can thus be very significant. The quality of the image is obviously related to the accuracy of the numerical method, which justifies the development of fast and accurate solvers for large problems. The size of the discrete system is an important issue but it is not the only one. Indeed, it is necessary to consider real propagation domains, which means that multiscale problems must be solved. Propagation domains are mostly wide, while the parameters that characterize the medium vary quickly. As a consequence, the representation of the parameters should be done on a fine grid while it should be sufficient to cover the medium with a coarse mesh. Obviously, it is possible to do computations with a fine mesh whose dimensions are fixed by the physical

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parameters. But, in that case, the resulting discrete system will contain a huge number of discrete unknowns, so that computational costs of the RTM become prohibitive, knowing that several solutions of the wave equation are needed. A numerical method capable of considering these two scales independently is thus of great interest, in particular for numerical geophysics.

To tackle multiscale problems, different attempts have been proposed in the literature. They involve upscaling, which consists in defining equivalent parameters. There exist many techniques of upscaling that are based on averaging or renormalizing the parameters [1,2]. Homogenization can also be applied [3–5]. It allows to get an accurate solution computed on the coarse grid without computing the full solution inside the fine mesh. It leads then to constant equivalent parameters. Now, it is worth noting that homogenization assumes that the parameters vary into different scales and that the medium is periodic.

In this paper, we focus on an operator-based upscaling method which can be applied without assuming periodicity of the medium. Operator-based upscaling methods were first developed for elliptic flow problems (see [6,7]) and then extended to hyperbolic problems (see [8–10]) such as the wave equation. The operator-based upscaling method is based on the splitting of the solution into two parts, the so-called rough and refined parts. The rough component is computed on a coarse grid while the refined component is obtained from computations on a fine mesh covering each coarse cell. The time computational costs can then be reduced by making calculations inside each coarse cell independent. This can be done by enforcing a Dirichlet boundary condition on the boundary of each coarse cell. By this way, the refined component is computed by solving local problems while the rough component is obtained classically by solving the variational problem inside the coarse mesh.

Operator-based upscaling methods were so far developed by using continuous finite elements. For instance, they have been carried out for wave problems by using mixed finite elements [8–10].

Herein, we consider the interest of developing an operator-based upscaling method using a Discontinuous Galerkin Finite Element Method (DGFEM). By this way, we would like to know whether it is possible to reduce the computational costs even more. DGFEMs perform well in the case of heterogeneous media because they match with hp-adaptivity and parallel computing. Nevertheless, for the same mesh, they involve more degrees of freedom than continuous FEMs. Hence, it would be interesting to see if, combined with an operator-based upscaling, it is possible to reduce the computational costs.

This study is preliminary to a work dealing with harmonic wave equations and thus concerns the Laplace operator. We consider the standard Laplace problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is the unit square $]0, 1[\times]0, 1[$ and the source term f lies in $L^2(\Omega)$. For the sake of simplicity, we restrict ourselves to a square domain, but the study can be extended to any convex polygonal domain.

Since it is known for being both stable and consistent, we are interested in the Interior Penalty Discontinuous Galerkin Method (IPDGM) [11]. We have organized the paper in such a way that we first show how to do upscaling with an IPDGM and next we compare the performances of this approach with the one involving a continuous Finite Element Method (FEM). In Section 2, we detail the variational framework to perform upscaling with the FEM and the IPDGM. The matrices resulting from the upscaling discretization are clarified in Section 3. Section 4 is devoted to detail the upscaling algorithm and a discussion on its performances. Finally, in Section 5, we compare the performances of the algorithms using the FEM and the IPDGM through numerical results.

2. Continuous and discontinuous finite element methods for upscaling

The *upscaling* method consists in finding a finite element solution in a space $V_{H,h}$ that is decomposed into the direct sum $V_{H,h} = V_H \oplus \hat{V}_{H,h}$. The space V_H is defined on a coarse grid of characteristic length H partitioning the domain Ω , and $\hat{V}_{H,h}$ is defined on a fine grid of characteristic length h which is obtained by refining the coarse grid. Then, the approximate solution $u_{H,h}$ is obtained as $u_{H,h} = u_H + \hat{u}_{H,h}$, where $u_H \in V_H$ and $\hat{u}_{H,h} \in \hat{V}_{H,h}$. In the following, u_H is called the rough component while $\hat{u}_{H,h}$ stands for the refined component representing the small scale effects. The space $\hat{V}_{H,h}$ must be appropriately defined in such a way that $\hat{u}_{H,h}$ can be easily computed as a function of u_H . The linear system involves thus only u_H . The small scale effects are then included inside the modeling afterwards thanks to the relation between $\hat{u}_{H,h}$ and u_H .

In this section, we present two different choices for the spaces V_H and $\hat{V}_{H,h}$. The first one is adapted to continuous finite elements and has been proposed in [6,7]. The second one involves discontinuous finite elements which we would like to focus on. Herein, we consider the IPDGM. Once the spaces are introduced, we define the corresponding bilinear forms.

2.1. Definition of the finite element spaces

Before introducing the finite element spaces, we need to define a partition of the domain Ω . For the sake of simplicity, we restrict ourselves to regular Cartesian meshes in two dimensions, but our study can be extended without difficulty to irregular meshes or to three dimensional problems. We fix $N, M \in \mathbb{N}^*$, and we define two mesh sizes $H = 1/N$ (the coarse

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