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Scattered data interpolation by bivariate splines with higher approximation order☆

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ABSTRACT

Given a set of scattered data, we usually use a minimal energy method to find a Lagrange interpolation in a bivariate spline space over a triangulation of the scattered data locations. It is known that the approximation order of the minimal energy spline interpolation is only 2 in terms of the size of triangulation. To improve this order of approximation, we propose several new schemes in this paper. Mainly we follow the ideas of clamped cubic interpolatory splines and not-a-knot interpolatory splines in the univariate setting and extend them to the bivariate setting. In addition, instead of the energy functional of the second order, we propose to use higher order versions. We shall present some theoretical analysis as well as many numerical results to demonstrate that our new interpolation schemes indeed have a higher order of approximation than the classic minimal energy interpolatory spline.

1. Introduction

Suppose $V = \{v_i := (x_i, y_i),\ i = 1, \ldots, n\}$ is a set of scattered data locations lying in a domain $\Omega \subset \mathbf{R}^2$. Let $\triangle$ be a triangulation of the data locations. Let $\{z_i, i = 1, \ldots, n\}$ be given real values. We would like to construct a smooth function $s \in C^r(\Omega)$ with $r \geq 1$ such that

$$s(v_i) = z_i, \quad i = 1, \ldots, n. \tag{1.1}$$

We shall use the following polynomial spline space throughout the paper:

$$S_d^r(\triangle) := \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_d,\ \forall T \in \triangle\},$$

where $d > r$ is a given integer, $\mathcal{P}_d$ is the space of bivariate polynomials of degree d, and $\triangle$ is a triangulation of the given data locations. This problem is known as bivariate spline Lagrange interpolation problem. A classic solution to this problem is the so-called minimal energy method (cf., e.g. [1]) which finds the spline $s^* \in S_5^1(\triangle)$ satisfying (1.1) such that

$$E_2(s^*) = \min\{E_2(s) : s(v_i) = z_i, i = 1, \ldots, n, s \in S_5^1(\triangle)\},$$

where E_2 is the so-called thin-plate energy functional defined by

$$E_2(s) = \int_\Omega [s_{xx}^2 + 2s_{xy}^2 + s_{yy}^2]dxdy. \tag{1.2}$$

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Of course, there are other methods to do Lagrange interpolation without minimizing an energy functional (cf., e.g. [2–4]). For example, in [3], the researchers use C^1 splines based on Clough–Tocher triangulation to do interpolation and also point out that their Lagrange interpolation schemes possess the optimal approximation order. Indeed, they combined extra smoothness conditions, Clough–Tocher splitting technique, and interpolation conditions to locally determine the MDS (minimal determine set) around each triangle and then use them to fix all the remaining coefficients to obtain a bivariate interpolatory spline. Such a spline interpolation possesses the optimal order of approximation. As we use the minimal energy method to globally fix all the extra coefficients besides interpolatory conditions, the surface of interpolatory spline created by the minimal energy method has minimal variation and oscillations. However, the approximation order is not optimal. It is known that the approximation order of the interpolatory spline obtained by the minimal energy method is 2 in terms of the size of triangulation (cf. [5]). In [5], the researchers explained that the order of approximation will not increase even if the degree of spline functions is increased. A numerical experiment is provided to show that the order is only 2 for various degrees.

How to increase the approximation order of the minimal energy method when doing scattered data interpolation is the main motivation of this paper. One approach is to interpolate derivative values in addition to function values. In [6], bivariate Hermite interpolatory splines were studied. The authors of their paper [6] established the approximation order of the bivariate spline Hermite interpolation scheme. The approximation order is indeed increased. See Theorem 2.2 in Section 2 for a special case $m = 3$. More precisely, for any integer $m \geq 2$, let

$$E_m(f) = \int_\Omega \left[\sum_{k=0}^{m} \binom{m}{k} \left[(D_x)^k (D_y)^{m-k} f \right]^2 \right] dxdy \tag{1.3}$$

be a general energy functional. A Hermite interpolatory spline $s^* \in S_d^{m-1}(\triangle)$ for an appropriate d, e.g. $d \geq 3m-1$ satisfying

$$D_x^\alpha D_y^\beta s^*(x_i, y_i) = f_{i,\alpha,\beta}, \quad \alpha + \beta \leq m-2, \tag{1.4}$$

such that

$$E_m(s^*) = \min\{E_m(s),\ s \in S_d^{m-1}(\triangle),\ s\text{satisfies (1.4)}\}.$$

Although such Hermite interpolatory splines have a higher order of approximation, in practice, we may not have these derivatives at all vertices or it needs a lot of effort and/or high cost to collect these derivative values.

The purpose of this paper is to construct several interpolatory spline schemes which achieve higher approximation order without using all derivative information. Recall as observed at the end of the paper [5], the error behavior is similar to the well-known natural cubic spline which minimizes the univariate energy $\int_a^b [s''(x)]^2 dx$ among all smooth functions that interpolate given values at points $a = x_0 < \cdots < x_n = b$. That is, numerical experiments show that the approximation of a minimal energy spline is better inside the underlying domain than near the boundary. It is also well-known that the full cubic spline space (with no special boundary conditions) has approximation power $\mathcal{O}(h^4)$ where h is the mesh size, but the interpolating natural spline only has approximation order $\mathcal{O}(h^2)$. This loss of accuracy is due to the natural boundary conditions, and indeed the interpolating spline does exhibit $\mathcal{O}(h^4)$ accuracy in a compact subset of $[a, b]$ which stays away from the boundary. Carl de Boor suggested that the analogous situation might also hold for bivariate minimal energy splines (cf. [5]). But the bivariate splines are much more complicated than univariate splines. There are too many extra coefficients besides the interpolatory conditions that need to be fixed in the bivariate spline setting. So we have to use the minimal energy method or other methods to solve this situation. Thus we propose the following new bivariate interpolatory spline schemes.

Clamped interpolation scheme: We find the spline function $s^* \in S_d^2(\triangle)$ satisfying the interpolation conditions (1.1) as well as boundary Hermite interpolation conditions

$$D_x^\alpha D_y^\beta s^*(x_i, y_i) = f_{i,\alpha,\beta}, \quad \alpha + \beta \leq 1,\ (x_i, y_i) \in \partial\Omega, \tag{1.5}$$

which minimizes E_3, where $\partial\Omega$ denotes the boundary of Ω. Here $d \geq 8$ if $\triangle$ is a general triangulation or appropriate d if $\triangle$ is a Cough–Tocher or Powell–Sabin refinement of triangulation or a FVS triangulation. See, e.g., [7].

Numerical experiments and a theoretical study show that the approximation order of this scheme is comparable to that of the bivariate Hermite spline interpolation discussed in [6], where the Hermite interpolatory splines use the derivatives at all vertices. Thus our clamped interpolatory splines are better in the sense that we use only derivatives at boundary vertices.

Again we face the challenge that we may not have boundary derivatives available for general real-life practical problems. Thus, we propose three different approaches to overcome this difficulty in Section 4.

Lagrange interpolation scheme with E_3: The easiest approach is to do nothing. That is, we find an $s^* \in S_8^2(\triangle)$ satisfying (1.1) which minimizes higher order energy functional E_3 instead of E_2. Our numerical experiments clearly show that Lagrange interpolation using E_3 is much better than that of using the thin-plate energy functional E_2.

Least squares scheme: For each boundary vertex v_b, we construct a least squares polynomial fitting to function values nearby a v_b and use its derivative to approximate the true derivatives at v_b. Then we use clamped spline interpolation discussed above. We will explain this approach in more detail in Section 4.2.