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Computation of integrals with oscillatory and singular integrands using Chebyshev expansions☆

Hongchao Kang, Shuhuang Xiang*, Guo He

School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, PR China

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ABSTRACT

We present a general method for computing oscillatory integrals of the form

$$\int_{-1}^{1} f(x)G(x)e^{i\omega x}dx,$$

where f is sufficiently smooth on $[-1, 1]$, ω is a positive parameter and G is a product of singular factors of algebraic or logarithmic type. Based on a Chebyshev expansion of f and the properties of Chebyshev polynomials, the proposed method for such integrals is constructed with the help of the expansion of the oscillatory factor $e^{i\omega x}$. Furthermore, due to numerically stable recurrence relations for the modified moments, the devised scheme can be employed to compute oscillatory integrals with algebraic or logarithmic singularities at the end or interior points of the interval of integration. Numerical examples are provided to confirm our analysis.

1. Introduction

In this paper we are concerned with the numerical evaluation of oscillatory integrals of the form

$$\int_{-1}^{1} f(x)G(x)e^{i\omega x}dx, \tag{1.1}$$

where f is a sufficiently smooth function on $[-1, 1]$, ω is a positive parameter and G is a product of singular factors of algebraic or logarithmic type. Here, weight functions $G(x)$ and parameters are listed in Table 1.

The first case of Table 1 corresponds to $\int_{-1}^{1} f(x)e^{i\omega x}dx$ and the integrals of the forms 2–5, 6, 8, 10–17 belong to the special cases of the form 18. The integrals 2–18 with the weak singularities arise in the numerical approximations of solutions to Volterra integral equations of the first kind involving highly oscillatory kernels with weak singularities [1,2]. In addition, it is well-known that the Radon transform, which plays an important role in the CT, PET and SPECT technology of medical sciences and is widely applicable to tomography, the creation of an image from the scattering data associated to cross-sectional scans of an object, is closely related to this form of oscillatory singular integrals [3,4]. The singularities in (1.1) are also called singularities of the Radon transform in medical tomography. Moreover, the integrals 2–18 are also used to the solution of the singular integral equation for classical crack problems in plane and antiplane elasticity [5]. Further, they can be taken as model integrals appearing in boundary integral equations for high-frequency acoustic scattering (e.g., high-frequency Helmholtz equation in two dimensions), where the kernels have algebraic or logarithmic singularities on the diagonal ([6,7] and references therein), which is also our main target application.

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* Corresponding author.
E-mail addresses: laokang834100@163.com (H. Kang), xiangsh@mail.csu.edu.cn (S. Xiang), beibingyang009@163.com (G. He).

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Table 1
Weight functions $G(x)$ and parameters.

Number c	$G(x)(G^{[c]}(x))$	Parameters
1	1	
2	$(1-x)^{\alpha}(1+x)^{\beta}$	$\alpha, \beta > -1$
3	$\ln(1-x)(1-x)^{\alpha}(1+x)^{\beta}$	$\alpha, \beta > -1$
4	$(1-x)^{\alpha}(1+x)^{\beta}\ln(1+x)$	$\alpha, \beta > -1$
5	$\ln(1-x)(1-x)^{\alpha}(1+x)^{\beta}\ln(1+x)$	$\alpha, \beta > -1$
6	$\lvert x-a\rvert^{\alpha}$	$\alpha > -1, \lvert a\rvert < 1$
7	$\lvert x-a\rvert^{\alpha}\operatorname{sgn}(x-a)$	$\alpha > -1, \lvert a\rvert < 1$
8	$\lvert x-a\rvert^{\alpha}\ln\lvert x-a\rvert$	$\alpha > -1, \lvert a\rvert < 1$
9	$\lvert x-a\rvert^{\alpha}\ln\lvert x-a\rvert\operatorname{sgn}(x-a)$	$\alpha > -1, \lvert a\rvert < 1$
10	$(1-x)^{\alpha}(1+x)^{\beta}\lvert x-a\rvert^{\gamma}$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
11	$(1-x)^{\alpha}(1+x)^{\beta}\lvert x-a\rvert^{\gamma}\ln\lvert x-a\rvert$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
12	$\ln(1-x)(1-x)^{\alpha}(1+x)^{\beta}\lvert x-a\rvert^{\gamma}$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
13	$(1-x)^{\alpha}(1+x)^{\beta}\ln(1+x)\lvert x-a\rvert^{\gamma}$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
14	$\ln(1-x)(1-x)^{\alpha}(1+x)^{\beta}\ln(1+x)\lvert x-a\rvert^{\gamma}$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
15	$(1-x)^{\alpha}(1+x)^{\beta}\ln(1+x)\lvert x-a\rvert^{\gamma}\ln\lvert x-a\rvert$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
16	$\ln(1-x)(1-x)^{\alpha}(1+x)^{\beta}\lvert x-a\rvert^{\gamma}\ln\lvert x-a\rvert$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
17	$\ln(1-x)(1-x)^{\alpha}(1+x)^{\beta}\ln(1+x)\lvert x-a\rvert^{\gamma}\ln\lvert x-a\rvert$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1$
18	$(\ln(1-x)^{s})^{p}(1-x)^{\alpha}(1+x)^{\beta}(\ln(1+x)^{t})^{q}\lvert x-a\rvert^{\gamma}(\ln\lvert x-a\rvert^{m})^{r}$	$\alpha, \beta, \gamma > -1, \lvert a\rvert < 1,$ $s, t, m \in \mathcal{R}, p, q, r = 0, 1, 2, \ldots$

The singularities of algebraic or logarithmic type and possible high oscillations of the integrands make the integrals (1.1) very difficult to approximate accurately using standard methods, e.g., Gauss–Legendre rule. It should be also pointed out that many efficient methods, such as the Filon-type [8,9], Levin-type [10] and numerical steepest descent methods [11], cannot be applied directly to the integrals of the forms 3–18 in Table 1, since the nonoscillatory parts of the integrands are undefined at the endpoints or one point in the interior of the interval. Moreover, for singular weight functions $G(x)$ of the forms 3–18 in Table 1, their moments may be awkward to compute. For example, for all $\alpha, \beta, \gamma > -1, |a| < 1, s, t, m \in \mathcal{R}, p, q, r = 0, 1, 2, \ldots$, neither the moments

$$\int_{-1}^{1} x^k (\ln(1-x)^s)^p (1-x)^{\alpha}(1+x)^{\beta}(\ln(1+x)^t)^q |x-a|^{\gamma}(\ln|x-a|^m)^r e^{i\omega x}dx, \tag{1.2}$$

nor the Chebyshev moments

$$\int_{-1}^{1} T_k(x)(\ln(1-x)^s)^p (1-x)^{\alpha}(1+x)^{\beta}(\ln(1+x)^t)^q |x-a|^{\gamma}(\ln|x-a|^m)^r e^{i\omega x}dx, \tag{1.3}$$

is often easily computed, where $T_k(x)$ is the Chebyshev polynomial of the first kind of degree k. Unfortunately, since both the Filon-type method [8,9] and the Clenshaw–Curtis-type method [12] require that the above two moments (1.2)–(1.3) can be easily calculated, they are unavailable for the integrals of the forms 3–18 in Table 1. For the integrals (1.1), the method proposed in this paper first expands f into a series of Chebyshev polynomials, based on the ideas presented in [13]. Then, the factor $e^{i\omega x}$ is expanded into a series of Chebyshev polynomials and Bessel functions of the first kind. And, thanks to the properties of Chebyshev polynomials, we extend, in a sense, the ideas presented in [14], where the recurrence formula for the calculation of the modified moments

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} T_n(x)dx, \tag{1.4}$$

was presented. It is worth noting that the modified moments (1.4) play an important role in the construction of quadrature formulas for the integrals (1.1). Furthermore, all the required moments $\int_{-1}^{1} T_k(x)G(x)e^{i\omega x}dx$, are expressed in terms of the simpler ones

$$\int_{-1}^{1} T_k(x)G(x)dx, \tag{1.5}$$

which can be computed efficiently. What is more important, the moments (1.5) satisfy recurrence relations that are stable in either forward or backward direction, which makes the whole algorithm quite simple. Hence, the presented method allows us to overcome some problems that usually appear in the case when the integrands are both singular and oscillating.

Here, we would also like to mention several other papers related to the ideas given in this article. In [15,16], the algorithms for computing integrals of the form $\int_{-1}^{1} f(x)K(x)dx$ were proposed for a few simple singular/oscillating functions K. As in this paper, the algorithms mentioned in [15,16], are based on expanding the function f into a series of Chebyshev polynomials. The methods presented in [15,16] were constructed only for four simple functions $K(x)$, but in [17,18] the idea was extended and generalized for a wide class of oscillatory or singular integrands. In [19,20], two different approaches based