Journal of Computational and Applied Mathematics 242 (2013) 213–231

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On the numerical treatment of linear–quadratic optimal control problems for general linear time-varying differential-algebraic equations[☆]

Stephen L. Campbell[a], Peter Kunkel[b,*]

[a] Department of Mathematics, Box 8205, North Carolina State University, Raleigh, NC 27695-8205, USA
[b] Mathematisches Institut, Universität Leipzig, Augustusplatz 10, 04109 Leipzig, Federal Republic of Germany

ARTICLE INFO

Article history:
Received 8 February 2012
Received in revised form 12 October 2012

MSC:
49J15
49M05
49M25
65L80

Keywords:
Differential-algebraic equation
Optimal control
Radau
Gauß–Lobatto
Direct transcription
Numerical methods

ABSTRACT

The development of numerical methods for finding optimal solutions of control problems modeled by differential-algebraic equations (DAEs) is an important task. Usually restrictions are placed on the DAE such as being semi-explicit. Here the numerical solution of optimal control problems with linear time-varying DAEs as the process and quadratic cost functionals is considered. The leading coefficient is allowed to be time-varying and the DAE may be of higher index. Both a direct transcription approach and the solution of the necessary conditions are examined for two important discretizations.

1. Introduction

The Linear–Quadratic Regulator (LQR) problem is an important problem in control theory that arises in a number of situations including tracking problems and robust control. The LQR problem is also important as a test problem.

Differential-algebraic equations (DAEs) appear in a wide variety of areas ranging from chemical engineering to mechanics to electrical systems [1,2]. The control of DAEs has naturally received considerable study but has usually been limited to special classes of DAEs [3–5].

Linear time-varying systems pose a number of challenges. They arise naturally and as the linearizations of nonlinear systems. The theory for general linear time-varying DAEs is considerably more complicated than that for linear time-invariant or semi-explicit DAEs.

There has been considerable work on the interrelated problems of numerical optimal control by different approaches including direct transcription, differential-algebraic equations, and boundary value problems. We note only [6–20]. In this paper we consider the numerical solution of LQR problems with linear time-varying DAE processes. This paper is unique in

☆ This work has been supported by the *Deutsche Forschungsgemeinschaft* under grant no. KU964/7-1 and in part by NSF Grant DMS-0907832.
* Corresponding author.
E-mail addresses: slc@math.ncsu.edu (S.L. Campbell), kunkel@math.uni-leipzig.de (P. Kunkel).

0377-0427/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2012.10.011

several ways. A major one is that we consider higher index DAEs and we allow the leading coefficient to be variable. Thus we do not assume that the system is semi-explicit which is often done in the numerical optimal control literature; see [3].

In particular, in this paper we discuss various possibilities for the numerical treatment of linear–quadratic optimal control problems consisting of a quadratic cost functional (omitting for simplicity the argument in all coefficient functions)

$$\mathcal{J}(x,u) = \frac{1}{2}x(\bar{t})^T M x(\bar{t}) + \frac{1}{2}\int_{\underline{t}}^{\bar{t}} x^T W x + 2x^T S u + u^T R u \, dt, \tag{1}$$

where $W \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $S \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $R \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$, $M \in \mathbb{R}^{n,n}$, $\mathbb{I} = [\underline{t}, \bar{t}]$, and an initial value problem for a linear DAE

$$E\dot{x} = Ax + Bu + f, \quad x(\underline{t}) = \underline{x} \tag{2}$$

as constraint, where $E \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $A \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $f \in C^0(\mathbb{I}, \mathbb{R}^n)$, and $\underline{x} \in \mathbb{R}^n$. Without loss of generality, we assume that W and R are pointwise symmetric and that $M \in \mathbb{R}^{n,n}$ is symmetric. Additionally, we assume that all given functions are sufficiently smooth. We then look for a trajectory (x, u) with x in an appropriately chosen subspace $\mathbb{X} \subseteq C^0(\mathbb{I}, \mathbb{R}^n)$ and $u \in \mathbb{U} = C^0(\mathbb{I}, \mathbb{R}^m)$ which minimizes the cost (1).

We may distinguish between three main approaches for the determination of numerical approximations to the optimal trajectory (x, u). The first approach is to derive the corresponding necessary conditions for a local extremum. In the present case, these are given by a boundary-value problem for a DAE which includes an adjoint equation for the involved Lagrange multiplier. This boundary-value problem is then discretized to get the desired numerical approximations. For short, this approach is called "first optimize, then discretize". The second approach avoids involving the necessary conditions belonging to the given continuous problem by starting with the discretization of the whole optimal control problem. In this way, we get a finite-dimensional (discrete) optimal control problem for which the necessary conditions are well-known. For short, this approach is called "first discretize, then optimize". We refer to this as "direct transcription". In industrial grade direct transcription codes being applied to nonlinear problems the resulting finite dimensional problem is often solved by a sequential quadratic programming or a barrier method utilizing sparse linear algebra [3].

The third approach, which we will not discuss further here, is control parameterization. In control parameterization one chooses a family of controls which are finitely parameterized. Then integration of the dynamics and evaluation of the cost provides a function of the parameters which can be fed to a standard optimizer. Due to its ease of implementation on problems that are easy to integrate, control parameterization is also widely used in engineering. For control parameterization it is clear what is considered the control since the control is treated as known for each evaluation of the cost. However, in direct transcription and the first approach that is not really the case. There are variables that are differentiated and those that are not. The undifferentiated variables are called algebraic variables. In general the algebraic variables include u and part of x. What is important is not what the user thinks is the control but rather that there exists a choice of variables such that if they were the control, then an index one system results [21].

In the case of the problem studied here, standard techniques applied to the transcribed finite dimensional problem can yield the desired numerical approximations together with some discrete Lagrange multipliers. As noted direct transcription codes do not set up and solve necessary conditions. They use iterative optimization algorithms. However, in this paper we are analyzing two approaches so we will set up the necessary conditions for the transcribed problem for the purpose of analysis.

If the free DAE (that is setting $u = 0$) in the constraint is semi-explicit and of differentiation index one, direct transcription can be shown to yield a discretization of the necessary conditions belonging to the continuous problem [22]. In the present paper, we deal with the general case of a general (possibly higher-index) linear time-varying DAE. As discussed in [23], we replace this DAE by a DAE with differentiation index one which has the same solutions as the original DAE. This new DAE may not be semi-explicit. There are then two prominent families of one-step methods that are able to integrate such DAEs, namely the Radau IIa methods [24], and the Gauß–Lobatto methods [25]. We will show how they can be used to discretize the necessary conditions of the continuous problem and we will discuss what happens when we use them for direct transcription.

This paper is organized as follows. In Section 2, we state a correct formulation of the optimal control problem by performing an index reduction and list some properties that are important for the following discussion. In Section 3, we present possible discretizations of the necessary conditions of the continuous problem based on the Radau IIa and Gauß–Lobatto methods. We then discuss direct transcription based on the same methods in Section 4. In particular, we address the question of determining in what way the equations obtained by direct transcription can be interpreted as a discretization of the necessary conditions of the continuous problem. The different approaches are compared in Section 5 with the help of some numerical experiments. Finally, Section 6 gives some conclusions.

2. Preliminaries

As mentioned in the introduction, the first step in the treatment of the given optimal control problem is to perform an index reduction for the DAE. This index reduction is based on an appropriate assumption on the regularity of the given DAE. Following [23], we assume the following Hypothesis 1, which is based on a so-called behavior formulation

$$\mathcal{E}\dot{z} = \mathcal{A}z + f, \tag{3}$$