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Computational survey on a posteriori error estimators for nonconforming finite element methods for the Poisson problem[☆]

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ABSTRACT

This paper compares different a posteriori error estimators for nonconforming first-order Crouzeix–Raviart finite element methods for simple second-order partial differential equations. All suggested error estimators yield a guaranteed upper bound of the discrete energy error up to oscillation terms with explicit constants. Novel equilibration techniques and an improved interpolation operator for the design of conforming approximations of the discrete nonconforming finite element solution perform very well in an error estimator competition with six benchmark examples.

1. Introduction

The a posteriori error analysis of conforming FEM is well established and contained even in textbooks [1–5]. Although a unified framework is established [6], much less is known about a posteriori error analysis for nonconforming lowest-order Crouzeix–Raviart finite element methods [7–15].

The Helmholtz decomposition allows a split of the error in the broken energy norm

$$|||e|||_{NC}^2 \leqslant \eta^2 + |||\mathrm{Res}_{NC}|||_\star^2.$$

The first term η on the right-hand side involves contributions of the data f and is directly computable (up to quadrature errors); cf. (3.1) for an explicit representation. The second term $|||\mathrm{Res}_{NC}|||_\star$ in the upper error bound is the weighted dual norm of some residual which can indeed be estimated by a posteriori error estimators for Poisson problems such as equilibration error estimators [2,3,16–19], least-squares error estimators [5] or localisation error estimators [20]; another class of possible estimators exploits the identity

$$|||\mathrm{Res}_{NC}|||_\star = \min_{\substack{v\in H^1(\Omega)\\ v=u_D \text{ on } \partial\Omega}} \|\varkappa^{1/2}(\nabla_{NC} u_{CR} - \nabla v)\|_{L^2(\Omega)}$$

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Table 1
Benchmark Poisson examples and subsection references.

Ref.	Short name	Problem data	Feature
7.1	L-shaped domain	$f \equiv 0$, $u_D \neq 0$, $\varkappa \equiv 1$	Corner singularity
7.2	Slit domain	$f \equiv 1$, $u_D \neq 0$, $\varkappa \equiv 1$	Slit singularity
7.3	Square domain	$f \notin P_0(\mathcal{T})$, $u_D \equiv 0$, $\varkappa \equiv 1$	Oscillations
7.4	3/4-Disk domain	$f \notin P_0(\mathcal{T})$, $u_D \equiv 0$, $\varkappa \equiv 1$	Osc. and corner sing.
7.5	Square domain	$f \equiv 0$, $u_D \neq 0$, $\varkappa \neq 1$	Diffusion jumps
7.6	Octagon domain	$f \equiv 0$, $u_D \neq 0$, $\varkappa \neq 1$	Diffusion jumps

Table 2
Classes of a posteriori error estimators in this paper.

No	Classes of error estimators	Class representatives
1	Interpolation	η_{A}, $\eta_{\mathrm{MP1RED(0)}}$, η_{PMRED}, η_{AP2}
2	Minimisation	η_{MP1}, $\eta_{\mathrm{MP1RED(k)}}$, η_{MP2}
3	Equilibration	η_{B}, η_{LW}
4	Least-squares	η_{Repin}
5	Localisation	η_{CF}

of [6,12] and Theorem 3.1b. Those upper bounds of $|||\mathrm{Res}_{\mathrm{NC}}|||_\star$ compute some test functions $v_{\mathrm{xyz}} \in H^1(\Omega)$ with $u = u_D$ on $\partial\Omega$ and evaluate

$$|||\mathrm{Res}_{\mathrm{NC}}|||_\star \leqslant \|\varkappa^{1/2}(\nabla_{\mathrm{NC}} u_{\mathrm{CR}} - \nabla v_{\mathrm{xyz}})\|_{L^2(\Omega)}.$$

Three explicit designs in Sections 4.1–4.2 provide estimators from μ_{A} after [12] and μ_{AP2} after [15,21,22], plus novel error estimators $\mu_{\mathrm{MP1RED(0)}}$ and μ_{PMRED} while global minimisation in some discrete subspace leads in Section 4.3 to μ_{MP1}, μ_{MP1RED} and μ_{MP2}.

This paper concerns the Poisson model interface problem: Given a right-hand side $f \in L^2(\Omega)$, the Dirichlet data $u_D \in H^1(\Omega)$ and some bounded, piecewise constant diffusion coefficient

$$0 < \underline{\varkappa} \leqslant \varkappa(x) \leqslant \overline{\varkappa} < \infty \quad \text{for a.e. } x \in \Omega \tag{1.1}$$

in the domain Ω, seek $u \in H^1(\Omega)$ with

$$-\operatorname{div}(\varkappa\nabla u) = f \quad \text{in } \Omega \quad \text{and} \quad u = u_D \quad \text{on } \partial\Omega. \tag{1.2}$$

The primal variable u will be discretised with nonconforming Crouzeix–Raviart FEMs on some regular triangulation $\mathcal{T}$ of Ω into triangles.

In this paper, the a posteriori error estimators of Table 2 compete in the 6 benchmark problems of Table 1. The 11 error estimators also give rise to adaptive mesh-refinement strategies with the overall experience that all lead to comparable mesh refinement that recovers the optimal convergence rate. Numerical evidence supports the superiority of the novel error estimator η_{PMRED} from Section 4.2 and η_{AP2} for adaptive a posteriori error control with efficiency indices in the range of 1.2–1.5. Since the overhead by η leads to only little overestimation of around 15%, it is indeed worth to utilise a more costly and more accurate evaluation of $|||\mathrm{Res}|||_\star$. In examples with constant coefficients, three iterations of some preconditioned conjugated gradient scheme with initial value $\mu_{\mathrm{MP1RED(0)}}$ leads to a cheap and highly efficient error estimator $\mu_{\mathrm{MP1RED(3)}}$ close to the optimum $\mu_{\mathrm{MP1RED(\infty)}}$; in examples with discontinuous coefficients the improvement after three iterations is less significant.

The remaining parts of this paper are outlined as follows. Section 2 introduces the necessary notation and preliminaries. Section 3 presents the a posteriori error analysis. Section 4 gives details on the realisations of upper bounds of $|||\mathrm{Res}_{\mathrm{NC}}|||_\star$. Section 5 deals with modifications in case of inhomogeneous boundary conditions. The novel application of equilibration techniques for a posteriori error control of nonconforming finite element methods is introduced in Section 6. In Section 7 all estimators of Table 2 are compared with the six benchmark problems from Table 1. Section 8 draws some conclusions on the numerical experiments and adds some overall remarks.

2. Notation and preliminaries

2.1. Crouzeix–Raviart finite element spaces

Given a regular triangulation $\mathcal{T}$ of the bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, into triangles with edges $\mathcal{E}$, nodes $\mathcal{N}$ and free nodes $\mathcal{M}$, the midpoints of all edges are denoted by $\mathrm{mid}(\mathcal{E}) := \{\mathrm{mid}(E) \mid E \in \mathcal{E}\}$ and the boundary edges along $\partial\Omega$ are denoted by $\mathcal{E}(\partial\Omega) := \{E \in \mathcal{E} \mid E \subseteq \partial\Omega\}$ while $\mathcal{E}(\Omega) := \mathcal{E} \setminus \mathcal{E}(\partial\Omega)$.