Journal of Computational and Applied Mathematics 245 (2013) 86–96

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Linearizability and local bifurcation of critical periods in a cubic Kolmogorov system

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ARTICLE INFO

Article history:
Received 17 January 2012
Received in revised form 19 August 2012

MSC:
34C05
34C23
68-04

Keywords:
Kolmogorov system
Center
Isochronicity
Integrability
Bifurcation

ABSTRACT

Since Chicone and Jacobs investigated local bifurcation of critical periods for quadratic systems and Newtonian systems in 1989, great attention has been paid to some particular forms of cubic systems having special practical significance but less difficulties in computation. This paper is devoted to the linearizability and local bifurcation of critical periods for a cubic Kolmogorov system. We use the Darboux method to give explicit linearizing transformations for isochronous centers. Investigating the finite generation for the ideal of all period constants, which are of the polynomial form in six parameters, we prove that at most two critical periods can be bifurcated from the interior equilibrium if it is an isochronous center. Moreover, we prove that the maximum number of critical periods is reachable.

1. Introduction

Consider the planar differential system

$$\dot{x} = F(x, y), \qquad \dot{y} = G(x, y),$$

where functions F and G are analytic at the origin $O : (0, 0)$ and satisfy $F(0, 0) = G(0, 0) = 0$. The origin O is a nondegenerate equilibrium if the Jacobian matrix $\frac{\partial(F,G)}{\partial(x,y)}$ has no vanished eigenvalues at O. In this case topological properties of all orbits near O are completely determined unless the matrix has a pair of pure imaginary eigenvalues. The equilibrium O with a pair of pure imaginary eigenvalues is called a *center-focus*, to which a further identification of center from focus, called the *center problem*, is of special interest. Consider the polynomial system

$$\dot{x} = -y + \sum_{j=2}^{n} F_j(x, y), \qquad \dot{y} = x + \sum_{j=2}^{n} G_j(x, y), \tag{1.1}$$

where $F_j(x, y)$ and $G_j(x, y)$ are both homogeneous of degree j in x and y. For $n = 2$ the center conditions are surveyed in [1–3]. For $n = 3$, all center conditions were given by Sibirski [4] in the homogeneous case, i.e., $F_2(x, y) \equiv 0$ and $G_2(x, y) \equiv 0$

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doi:10.1016/j.cam.2012.12.003

in system (1.1). Concerning centers of nonhomogeneous cubic systems, the cubic Liénard system was discussed in [1,5], quadratic-like cubic systems in [6] and cubic Kukles system in [7,8].

A further investigation to centers is concerning *isochronicity*, i.e., showing whether those periodic orbits near the center are synchronous or, equivalently say, have the same period. The research of isochronicity can be recalled up to 17th century when Dutch mathematician Christian Huygens investigated the cycloidal pendulum, much earlier than the development of differential calculus, and has been an attractive problem [9] since then. In the sense of normal forms, isochronicity is proved to be equivalent to analytical linearization at the center (see e.g. [3,10,11]). So the isochronicity problem is also called the *linearizability problem* at centers. All conditions of linearizability were given by Loud [12] for $n = 2$, by Pleshkan [13] for system (1.1) with homogeneous cubic nonlinearities, by Lloyd et al. [6] and Chavarriga et al. [14] for cubic systems with degenerate infinity and by Romanovski et al. [15] for system (1.1) with homogeneous quintic nonlinearities. Some results on linearizability were also obtained for time-reversible systems [16–18]. In contrast, for those non-isochronous centers, called weak center, it is also interesting to study the local bifurcation of critical periods when the parameters are perturbed in the center variety, the theory of which was founded by Chicone and Jacobs [10] in 1989. From then on, the local bifurcation of critical periods is investigated for many systems such as cubic systems with homogeneous nonlinearities [19], reduced cubic Kukles system [20], cubic Liénard equations [21] and generalized Lotka–Volterra systems [22].

The Kolmogorov system

$$\dot{x} = xf(x, y), \qquad \dot{y} = yg(x, y) \tag{1.2}$$

is a significant model to describe the interaction of two species occupying the same ecological niche. One of the most important things is to discuss qualitative properties of its interior equilibria and periodic orbits nearby, giving states of permanence. The quadratic Kolmogorov system, i.e., $f(x, y)$ and $g(x, y)$ in (1.2) are both linear, is called the Lotka–Volterra–Gause model, all possible phase portraits are plotted in [23], showing that periodic orbits can only appear near a center. Furthermore, Waldvogel [24] proved that there are no isochronous centers in the quadratic Kolmogorov system. For the cubic Kolmogorov system, i.e., $f(x, y)$ and $g(x, y)$ in (1.2) are both quadratic polynomials, few results were given for conditions of centers and linearizability except for [25], in which an one-parameter family of cubic Kolmogorov systems is shown to have an isochronous center by the Urabe's Criteria. The main difficulties come from the normalized system, which has the standard linear term $(-y, x)$ but the coefficients of higher order terms are functions of original parameters in much more complicated relations than quadratic ones.

In this paper we consider the cubic Kolmogorov system

$$\begin{cases} \dot{x} = x(A_1(x-a) + A_2(y-b) + B_3(x-a)^2 + B_4(y-b)^2) = P(x, y), \\ \dot{y} = y(B_1(x-a) + B_2(y-b) + B_3(x-a)^2 + B_4(y-b)^2) = Q(x, y), \end{cases} \tag{1.3}$$

where six parameters $A_1, A_2, B_1, \dots, B_4 \in \mathbb{R}$ are included. It possesses a more general form than the one considered in [25]. From the results of [6] we exhibit all conditions of center at the nondegenerate interior equilibrium $S : (a, b)$, where $ab \neq 0$. Then, we use the Darboux method [11] to find explicit linearizing transformations and obtain all conditions of linearizability at the center, which generalize the results of [25]. Furthermore, we show that at most two critical periods can be bifurcated near the interior equilibrium if is a weak center. In the case of isochronicity, finding the minimal ideal generated by all period constants, which are actually polynomials of the six parameters, we prove that also at most two critical periods can be bifurcated. Finally, proving the independence among those generators of the minimal ideal, we show that the maximum number of critical periods is reachable.

2. Centers and linearizability

Consider the cubic Kolmogorov system (1.3). One can compute

$$\left.\frac{\partial(P, Q)}{\partial(x, y)}\right|_{(x,y)=(a,b)} = \begin{pmatrix} aA_1 & aA_2 \\ bB_1 & bB_2 \end{pmatrix}$$

and obtain the characteristic equation $\lambda^2 - (aA_1 + bB_2)\lambda + ab(A_1B_2 - A_2B_1) = 0$. It implies that the equilibrium S is a center-focus if and only if $(aA_1 + bB_2)^2 - 4ab(A_1B_2 - A_2B_1) < 0$ and $aA_1 + bB_2 = 0$. Since we consider system (1.3) with a center-focus at S, in what follows we discuss the system

$$\begin{cases} \dot{x} = x\left(-\dfrac{bB_2}{a}(x-a) + A_2(y-b) + B_3(x-a)^2 + B_4(y-b)^2\right), \\ \dot{y} = y(B_1(x-a) + B_2(y-b) + B_3(x-a)^2 + B_4(y-b)^2), \end{cases} \tag{2.1}$$

where $a, b, A_2, B_1, \dots, B_4 \in \mathbb{R}$, $ab \neq 0$ and $b^2B_2^2 + abA_2B_1 < 0$. Applying the transformation

$$u = \frac{x}{a} - 1, \qquad v = \frac{y}{b} - 1,$$