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On the alternating direction method of multipliers for nonnegative inverse eigenvalue problems with partial eigendata

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ABSTRACT

We consider the nonnegative inverse eigenvalue problem with partial eigendata, which aims to find a nonnegative matrix such that it is nearest to a pre-estimated nonnegative matrix and satisfies the prescribed eigendata. In this paper, we propose several iterative schemes based on the alternating direction method of multipliers for solving the nonnegative inverse problem. We also extend our schemes to the symmetric case and the cases of prescribed lower bounds and of prescribed entries. Numerical tests (including a practical engineering application in vibrations) show the efficiency of the proposed iterative schemes.

1. Introduction

An n-by-n matrix $C \geq 0$ ($C > 0$, respectively) is called nonnegative (positive, respectively) if all its entries are greater than or equal to zero (greater than zero, respectively). Nonnegative matrices arise in various applications including game theory, Markov chain, probabilistic algorithms, numerical analysis, discrete distributions, categorical data, group theory, matrix scaling, economics, etc. For the applications and mathematical properties of nonnegative matrices, one may refer to [1–4] and references therein.

The nonnegative inverse eigenvalue problem aims to reconstruct a nonnegative matrix from its spectrum or partial eigendata. The nonnegative inverse eigenvalue problem has got much attention since 1940s (see for instance [5,6] and references therein) and many authors considered its solvability based on the complete set of eigenvalues. Recently, a few numerical algorithms were developed for computational purpose, including the isospectral flow method [7–10], the alternating projection method [11], and the nonsmooth Newton-type method [12]. In particular, the isospectral flow method in [7] was extended to the case of prescribed structures and the nonsmooth Newton-type method in [12] was extended to the cases of lower bounds and of prescribed entries.

In this paper, we consider the following nonnegative inverse eigenvalue problem (NIEP): Given a predetermined n-by-n nonnegative matrix C_o and a self-conjugate set of partial measured eigendata $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^{p}$ with $\lambda_j \in \mathbb{C}$, $\mathbf{x}_j \in \mathbb{C}^n$, and $p \ll n$, find an n-by-n nonnegative matrix C such that it is nearest to a pre-estimated n-by-n nonnegative matrix C_o in the Frobenius norm and has $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^{p}$ as its eigenpairs.

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In many applications, the entries of a nonnegative matrix stand for the physical parameters such as mass, length, elasticity, inductance, capacitance, and etc. (see for instance [13,14]). In practice, a predetermined nonnegative matrix C_o can be obtained from the real structure. However, the predicted dynamical behaviors by C_o, i.e., the eigenvalues and eigenvectors of C_o, often disagree with the experimentally measured data [13]. The nonnegative inverse problem aims to reconstruct a nonnegative matrix C from the measured eigendata.

We note that the NIEP is to find a solution of the following minimization problem.

$$\begin{aligned}
&\min\ \frac{1}{2}\|C-C_o\|^2 \\
&\text{subject to (s.t.) } CX = X\Lambda, \\
&C \geq 0,
\end{aligned} \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean vector norm or the Frobenius matrix norm and $(\Lambda, X) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}$ is the real matrix form of the prescribed self-conjugate eigendata $\{(\lambda_j, \mathbf{x}_j)\}_{j=1}^p$ as in [12]. For simplicity, we refer to the minimization problem (1) as the NIEP.

It is obvious that the NIEP is a convex quadratic programming problem, which can be solved by typical quadratic programming solvers based on interior point method (see for instance [15,16]). However, in practice, the problem size n is very large (say, $n \geq 1000$). Moreover, the number of equation constraints is also very large even when p is very small (say, if $p = 30$ but $n \geq 1000$, then $np \geq 30\,000$ or if $p = 200$ but $n \geq 1000$, then $np \geq 200\,000$). In this case, Newton-like or interior point algorithms are not so efficient for solving the NIEP since, in each iteration, solving a large-scale Newton equation whose dimensionality is proportional to np is inevitable.

In this paper, we propose several iterative schemes based on the alternating direction method of multipliers (ADMM) for solving the NIEP. This is motivated by the ADMM-based methods for variational problems [17–19] and semidefinite programming [20]. In particular, we present the original ADMM [18,21], an ADMM-based descent method [19], and an relaxed ADMM [22] for the NIEP. The main advantages of ADMM-based methods for the NIEP lie in that: Unlike Newton-like or interior point algorithms, the proposed ADMM-based methods reduce the problem complexity in the sense that no system of linear equations is necessary to solve; The included subproblems are easy to solve: A subproblem has a closed-form solution. The other subproblem is a standard equality-constrained quadratic programming but its solution is explicitly expressed based on the Moore–Penrose inverse of the matrix X of the measured eigenvectors, which can be computed with lower computation cost under a certain assumption on the eigendata. We also extend these ADMM-based schemes to the symmetric case and the cases of prescribed lower bounds and of prescribed entries. Finally, we report some numerical tests (including a practical engineering application in vibrations) to illustrate the efficiency of our methods.

Throughout the paper, we use the following notations. The symbols A^T, A^H, and $A^\dagger$ denote the transpose and the conjugate transpose, and the Moore–Penrose inverse of a matrix A, respectively. I is the identity matrix of appropriate dimension. Denote by $\|\cdot\|_{\max}$ the entry of largest absolute value of a matrix. Let $\mathbb{R}^{n\times n}$ and $\mathbb{SR}^{n\times n}$ be the set of all real matrices of order n and the set of all real symmetric matrices of order n, respectively. Let $\mathbb{R}_+^{n\times n}$ and $\mathbb{SR}_+^{n\times n}$ stand for the nonnegative orthants of $\mathbb{R}^{n\times n}$ and $\mathbb{SR}^{n\times n}$, respectively. Denote by $\Pi_{\mathcal{D}}\{\cdot\}$ the metric projection onto $\mathcal{D} \subseteq \mathbb{R}^{n\times n}$ (or $\mathbb{SR}^{n\times n}$).

The remainder of the paper is organized as follows. In Section 2 we present ADMM-based methods for solving the NIEP. In Section 3 we discuss some extensions. In Section 4 we report some numerical results to demonstrate the efficiency of the proposed methods.

2. Alternating direction method of multipliers

In this section, we present some ADMM-based Methods for solving the NIEP.

2.1. Reformulation

In this subsection, we give ADMM-oriented reformulation of the NIEP. Let $\mathbb{R}^{n\times n}$ ($\mathbb{SR}^{n\times n}$, respectively) be equipped with the inner product $\langle A_1, A_2\rangle = \mathrm{tr}(A_1^T A_2)$ for any $A_1, A_2 \in \mathbb{R}^{n\times n}$ ($\mathbb{SR}^{n\times n}$, respectively) and its induced norm $\|\cdot\|$, where "tr" means the trace of a matrix. Then the NIEP (1) can be reformulated as the following problem:

$$\begin{aligned}
&\min\ \frac{1}{2}\|C-C_o\|^2 + \frac{1}{2}\|Y-C_o\|^2 \\
&\text{s.t. } C - Y = 0, \\
&C \in \mathbb{R}_+^{n\times n}, \quad Y \in \mathcal{K},
\end{aligned} \tag{2}$$

where $\mathcal{K} := \{Y \in \mathbb{R}^{n\times n} : YX = X\Lambda\}$.

We note that Problem (2) is a convex minimization problem. Therefore, (C^*, Y^*) is a solution to Problem (2) if and only if there exists a point $Z^* \in \mathbb{R}^{n\times n}$ such that the following variational inequalities hold [15]

$$\begin{cases}
\langle C - C^*, C^* - C_o - Z^*\rangle \geq 0 \quad \forall C \in \mathbb{R}_+^{n\times n}, \\
\langle Y - Y^*, Y^* - C_o + Z^*\rangle \geq 0 \quad \forall Y \in \mathcal{K}, \\
\langle Z - Z^*, C^* - Y^*\rangle \geq 0 \quad \forall Z \in \mathbb{R}^{n\times n}.
\end{cases} \tag{3}$$