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Bases and dimensions of bivariate hierarchical tensor-product splines

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ABSTRACT

We prove that the dimension of bivariate tensor-product spline spaces of bi-degree (d, d) with maximum order of smoothness on a multi-cell domain (more precisely, on a set of cells from a tensor-product grid) is equal to the number of tensor-product B-spline basis functions, defined by only single knots in both directions, acting on the considered domain. A certain reasonable assumption on the configuration of the cells is required.

This result is then generalized to the case of piecewise polynomial spaces, with the same smoothness properties mentioned above, defined on a multi-grid multi-cell domain (more precisely, on a set of cells from a hierarchy of tensor-product grids). Again, a certain reasonable assumption regarding the configuration of cells is needed.

Finally, it is observed that this construction corresponds to the classical definition of hierarchical B-spline bases. This allows to conclude that this basis spans the full space of spline functions on multi-grid multi-cell domains under reasonable assumptions.

1. Introduction

1.1. Motivation and related works

Adaptive refinement of spline basis functions allows to localize changes in the control net so that the modification of a single control point will affect a limited region of the underlying geometric representation. Mesh refinement strategies constitute a fundamental component for the development of an effective approximation algorithm commonly used by standard surface reconstruction techniques. In the context of the numerical solution of partial differential equations, particular attention is currently devoted to this issue in connection with the emerging field of isogeometric analysis [1].

For this reason, refinement techniques which were originally introduced for standard geometric design applications, became the topic of recent studies, taking into account the dual requirements of geometry and analysis. The resulting novel perspective motivated new researches for the identification of geometric representations suitable for analysis which simultaneously satisfy the demand imposed by their use in the simulation framework and the accuracy of the geometrical model.

The extension of the isogeometric paradigm, originally introduced considering the NURBS model [2], with spline representations which allow local control of the refinement procedure has mainly focused on suitable applications [3–5] of the T-splines construction [6,7]. Subsequently, alternative solutions based on the so-called polynomial splines over T-meshes [8–10] and on hierarchical B-splines [11,12] have also been considered [13,14].

In this setting, the analytical point of view, which joins the geometric perspective, outlined the desire of characterizing the space spanned by the set of basis functions used to approximate the solution. This motivated investigations on the linear independence of T-splines blending functions [15,16], discussion about the dimension of related spline spaces [17] and the corresponding nested nature of these T-spline spaces [18].

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The hierarchical approach seems to be a valid solution to circumvent the weak points of T-splines identified by these studies (locality of the refinement [4], linear dependence associated with particular T-meshes [15], complexity of the enhanced refinement algorithm needed to ensure the linear independence of the blending functions [19]), and also the reduced regularity which is required for most results concerning splines over T-meshes. This includes the dimension results for spline spaces over T-meshes in the case that the degree is at least $2s + 1$ for splines with order of smoothness given by s that were derived in [8].

The hierarchical model allows complete control of the refinement by using a spline hierarchy whose levels identifies subsequent levels of refinement. We consider an *increasingly nested* sequence of tensor-product spline spaces $V^0 \subset V^1 \subset \cdots \subset V^{N-1}$, together with a *decreasingly nested* sequence of domains $\Omega^0 \supseteq \Omega^1 \supseteq \cdots \supseteq \Omega^{N-1}$. The cells of V^ℓ in $\Omega^\ell \setminus \Omega^{\ell+1}$ will be said to form a *multi-cell domain*. The union of these multi-cell domains will then be called a *multi-grid multi-cell domain*.

The simple idea of the hierarchical spline model is based on a suitable correlation between these two nested structures: at each level ℓ, for $\ell = 0, 1, \ldots, N-1$, we iteratively select the basis functions from the underlying spline space V^ℓ which act only on the current domain Ω^ℓ, i.e., whose support is contained in Ω^ℓ. At the same time we discard from the hierarchical basis the basis functions selected in the earlier steps which act only on Ω^ℓ. The local action of the refinement procedure is then immediately guaranteed by construction. Moreover, the local linear independence is inherited from the underlying B-spline bases.

The selection mechanism for the definition of a hierarchical B-spline basis introduced by Kraft [12] by means of subsequent dyadic refinements ensures that

- hierarchical basis functions allows proper local refinement and are linearly independent [12, Theorem 1],
- the hierarchical B-spline basis is *weakly* stable, i.e. the stability constants have at most a polynomial growth in the number of hierarchical levels [12, Theorem 3].

Hierarchical B-splines have already been applied in several applications related to geometric modeling — see for example [20–22]. In addition, a hierarchical quasi-interpolant together with approximation algorithms and scattered data approximation and interpolation problems were also discussed in [12]. A more detailed analysis of the above mentioned topics can be found in [23]. The case of partly overlapping boundaries of the sub-domains which are selected for further refinement and the application of hierarchical B-splines in isogeometric analysis have recently been considered in [14].

1.2. Contributions and outline

The goal of the present paper is to investigate dimensions and bases of hierarchical tensor-product B-spline spaces. The starting point of our study is a generalization of the dimension results for bivariate tensor-product polynomial spline spaces to multi-cell domains. When considering tensor-product spline functions with maximum order of smoothness, it turns out that the dimension formula on domains whose boundaries are piecewise linear curves (which satisfy a specific reasonable assumption) can be derived from the standard one related to rectangular grids (see, e.g., [24]) by including certain correction factors.

Under certain mild assumptions on the multi-cell domain considered at each level, the dimension of the above mentioned space is equal to the number of B-splines defined on the corresponding grid and which effectively act on it. This computation is then used to construct a basis for the space of bivariate tensor-product splines on multi-grid multi-cell domains, i.e., on hierarchies of multi-cell domains. This allows us to conclude that the classical hierarchical B-spline basis is indeed a basis for the considered spline space. *Consequently, the span of the hierarchical B-spline basis contains all spline functions of given bi-degree* (d, d) *and maximal smoothness that exist on the underlying hierarchical grid.*

The structure of the paper is as follows.

Section 2 reviews the notion of hierarchical B-splines and introduces tensor-product splines on multi-cell domains and on multi-grid multi-cell domains. It also provides a detailed overview of the subsequent analysis.

The next two sections study tensor-product splines on multi-cell domains. Section 3 derives the dimension result for the space of tensor-product splines of bi-degree (d, d) with maximum order of smoothness defined on a multi-cell domain. Subsequently, Section 4 identifies the assumptions concerning the cell configuration that are needed to guarantee that the set of B-splines which act on this set of cells is a basis for the spline space defined on it.

Section 5 uses these observations in order to obtain results on dimensions and bases for tensor-product splines, with the same smoothness properties as before, defined on multi-grid multi-cell domains. Finally, Section 6 concludes the paper.

2. Preliminaries

We start by revisiting the construction of hierarchical B-splines. We also describe the overall structure of the proposed analysis.

2.1. Hierarchical B-splines

Let $\{V^\ell\}_{\ell=0,\ldots,N-1}$ be a sequence of N nested tensor-product spline spaces so that