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On the convergence of a modified regularized Newton method for convex optimization with singular solutions

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ABSTRACT

In this paper we propose a modified regularized Newton method for convex minimization problems whose Hessian matrices may be singular. The proposed method is proved to converge globally if the gradient and Hessian of the objective function are Lipschitz continuous. Under the local error bound condition, we first show that the method converges quadratically, which implies that $\|x_k - x^*\|$ is equivalent to $\mathrm{dist}(x_k, X)$, where X is the solution set and $x_k \to x^* \in X$. Then we in turn prove the cubic convergence of the proposed method under the same local error bound condition, which is weaker than nonsingularity.

1. Introduction

Consider the unconstrained optimization problem

$$\min f(x), \quad x \in R^n, \tag{1.1}$$

where $f : R^n \to R$ is convex and twice continuously differentiable, whose gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$ are denoted by $g(x)$ and $G(x)$, respectively. Throughout the paper, we suppose that the solution set X of (1.1) is nonempty, and in all cases $\|\cdot\|$ stands for the 2-norm. It is clear that X is a closed convex set.

It is well-known that $f(x)$ is convex if and only if $G(x)$ is positive semidefinite for all $x \in R^n$. Moreover, if f is convex, then $x \in X$ if and only if x is a solution of the nonlinear equations

$$g(x) = 0. \tag{1.2}$$

There are many efficient methods [1–4] for solving the problem (1.1) or (1.2). The Newton method is one of the best known methods. At each iteration, the Newton method computes the trial step

$$d_k^N = -G_k^{-1} g_k,$$

where $g_k = g(x_k)$ and $G_k = G(x_k)$. An attractive feature of the Newton method is that it possesses quadratic convergence rate if $G(x^*)$ is nonsingular at a solution x^*, which implies that the solution is locally isolated.

However, the condition on the nonsingularity of the Hessian is too strong since many problems have singular solutions [5–7], which may contain some inverse problems and ill-posed problems [8]. To obtain reasonable solutions for

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this kind of problems, different regularization techniques are often used. Recently, under the local error bound condition, which is weaker than nonsingularity, Li et al. [6] proposed a regularized Newton method with quadratic convergence, where the trial step is the solution of the linear equations

$$(G_k + \lambda_k I)d = -g_k \quad \text{with } \lambda_k = C\|g_k\|$$

for some positive constant C, where I is the identity matrix. More details on the local error bound condition for nonlinear equations can be found in [5,7,9–12].

In this paper we propose a modified regularized Newton method for (1.1), which is mainly motivated in [9], where a modified Levenberg–Marquardt method was proposed for nonlinear equations with cubic convergence under the local error bound condition.

The main scheme of the modified regularized Newton method is given as follows. At each iteration, it solves the linear equations

$$(G_k + \lambda_k I)d = -g_k \tag{1.3}$$

to obtain the Newton step d_k, where λ_k is a suitable regularized parameter, and then solves the linear equations

$$(G_k + \lambda_k I)d = -g(y_k) \quad \text{with } y_k = x_k + d_k \tag{1.4}$$

to obtain the approximate Newton step $\hat{d}_k$.

The purpose of this paper is to investigate whether the proposed method has cubic convergence as the modified Levenberg–Marquardt method [9] under the local error bound condition.

The paper is organized as follows. In Section 2, we present the complete modified regularized Newton method carefully. In Section 3, we prove the global convergence of the proposed method. Quadratic convergence and cubic convergence of the proposed method are obtained in Section 4.

2. The algorithm

Let d_k and $\hat{d}_k$ be given by (1.3) and (1.4), respectively. Since the matrix $G_k + \lambda_k I$ is symmetric and positive definite, d_k is a descent direction of $f(x)$ at x_k, but $d_k + \hat{d}_k$ may not be. Hence we use a trust region technique to globalize the proposed method.

Let

$$\text{Ared}_k = f(x_k) - f(x_k + d_k + \hat{d}_k), \tag{2.1}$$

which is called the actual reduction of $f(x)$ at the k-th iteration.

Note that the Newton step d_k is the minimizer of the convex problem:

$$\min_{d\in R^n} \varphi_{k,1}(d) = \frac{1}{2}d^T G_k d + g_k^T d + \frac{1}{2}\lambda_k\|d\|^2. \tag{2.2}$$

If we let

$$\Delta_{k,1} = \|d_k\| = \|-(G_k + \lambda_k I)^{-1} g_k\|,$$

then it can be verified [2, Theorem 6.1.2] that d_k is also a solution of the trust region problem:

$$\min_{d\in R^n} \frac{1}{2}d^T G_k d + g_k^T d, \quad \text{s.t. } \|d\| \le \Delta_{k,1}.$$

By the famous result given by Powell in [13] (also see [2, Lemma 6.1.3]), we know that

$$\varphi_{k,1}(0) - \varphi_{k,1}(d_k) \ge \frac{1}{2}\|g_k\| \min\left\{\|d_k\|, \frac{\|g_k\|}{\|G_k\|}\right\}. \tag{2.3}$$

Similar to d_k, $\hat{d}_k$ is not only the minimizer of the problem:

$$\min_{d\in R^n} \varphi_{k,2}(d) = \frac{1}{2}d^T G_k d + g(y_k)^T d + \frac{1}{2}\lambda_k\|d\|^2, \tag{2.4}$$

but also the solution of the following trust region problem:

$$\min_{d\in R^n} \frac{1}{2}d^T G_k d + g(y_k)^T d, \quad \text{s.t. } \|d\| \le \Delta_{k,2},$$

where

$$\Delta_{k,2} = \|\hat{d}_k\| = \|-(G_k + \lambda_k I)^{-1} g(y_k)\|.$$