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Reliability analysis of a class of exponential distribution under record values[☆]

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ABSTRACT

In this paper the estimation of the parameters as well as survival and hazard functions for a class of an exponential family are presented by using Bayesian and non-Bayesian approaches under record values. Maximum likelihood and interval estimation are derived for the model parameters, and Bayes estimators of reliability performances are obtained under symmetric and asymmetric loss functions, when the parameters have discrete and continuous priors, respectively. Finally, two numerical examples are presented to illustrate the proposed results. An algorithm is introduced to generate record data, then the numerical example is given by using Monte Carlo simulation and different estimates results are compared.

1. Introduction

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with a cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. An observation X_j is called an upper record value if its value exceeds that of previous observations. Thus X_j is an upper record value if $X_j > X_i$ for each $j > i$. Then the record times sequence $\{T_n, n \geq 0\}$ is defined in the following manner

$$T_0 = 1 \quad \text{with probability } 1$$

and for $n \geq 1$,

$$T_n = \min\{j : X_j > X_{T_{n-1}}\}.$$

The sequence $\{X_{T_n}, n = 0, 1, 2, \ldots\}$ is called the sequence of upper records of the original sequence. An analogous definition can be obtained for lower records by changing the sign '$>$' into '$<$'. In this paper, we are only concerned with the upper record values, and similar results can be derived for lower records accordingly.

The concept of record values was first introduced by Chandler [1], which is a special order statistic from a sample whose size is determined by the values and the order of occurrence of observations. Record values are of interest and importance in many real life applications, such as weather, sports, economics, life-tests, stock market and so on. A growing interest in records has arisen in the last two decades and their properties have been extensively studied in the literature. See for example, [2,3]. For applications of the record values see [4]. More details about record values can be found in, for instance, [5,6].

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For the sake of simplicity, let $X = (X_{T_1}, X_{T_2}, \dots, X_{T_n}) = (X_1, X_2, \dots, X_n)$ be an upper record values with pdf $f(x)$ and cdf $F(x)$, $x = (x_1, x_2, \dots, x_n)$ is an observed value of X, then the joint pdf of X (see, e.g., [5,6]) can be expressed as

$$f(x) = f(x_n) \prod_{i=1}^{n-1} \frac{f(x_i)}{1 - F(x_i)}. \tag{1}$$

Consider random variable X has a continuous distribution function with cdf and pdf as follows

$$F(x; \beta, \theta) = 1 - \exp\{-\beta Q(x; \theta)\}, \quad 0 < x < \infty, \tag{2}$$

$$f(x; \beta, \theta) = \beta q(x; \theta) \exp\{-\beta Q(x; \theta)\}, \tag{3}$$

where β and θ are model parameters, $Q(x; \theta)$ is increasing in x with $Q(0; \theta) = 0$ and $Q(\infty; \theta) = \infty$, and the function $q(x; \theta) = (\partial/\partial x)Q(x; \theta) > 0$. The family in (2) is a general class of exponential-type distribution, and includes several well-known lifetime models such as exponential, Burr XII, Weibull, Pareto, Rayleigh and so on.

The survival function $R(t)$ and the hazard function $H(t)$ of the model (2) at mission time t can be written as

$$R(t) = \exp\{-\beta Q(t; \theta)\}, \qquad H(t) = \beta q(t; \theta). \tag{4}$$

Let $X_1 < X_2 < \cdots < X_n$ denote upper record values from the distribution with cdf (2), then from (1) the joint pdf of the records $X = (X_1, X_2, \dots, X_n)$ can be expressed as

$$L(\beta, \theta; x) = \beta^n \exp\{-\beta Q(x_n; \theta)\} \prod_{i=1}^{n} q(x_i; \theta). \tag{5}$$

The rest of the paper is organized as follows. Section 2 is devoted to give some frequentist estimation, such as maximum likelihood estimation (MLE) and interval estimation. Bayes analysis is presented in Section 3 for reliability performances under symmetric and asymmetric loss functions. Two numerical examples and some conclusions are given in Sections 4 and 5, respectively.

2. Frequentist estimation

In this section, the frequentist estimation for the model parameters is derived, which includes MLEs, as well as interval estimation of β and θ.

2.1. Point estimation

According to (5), the log-likelihood function can be written as

$$\ln L(\beta, \theta; x) = n \ln \beta - \beta Q(x_n; \theta) + \sum_{i=1}^{n} \ln q(x_i; \theta), \tag{6}$$

then we have

$$\frac{\partial}{\partial \beta} \ln L(\beta, \theta; x) = \frac{n}{\beta} - Q(x_n; \theta),$$

$$\frac{\partial}{\partial \theta} \ln L(\beta, \theta; x) = -\beta(\partial/\partial\theta)Q(x_n; \theta) + \sum_{i=1}^{n} \frac{(\partial/\partial\theta)q(x_i; \theta)}{q(x_i; \theta)}.$$

The maximum likelihood estimators of β and θ, namely $\hat{\beta}$ and $\hat{\theta}$, can be obtained by solving the following likelihood equations

$$\begin{aligned} &\beta = \frac{n}{Q(x_n; \theta)}, \\ &\sum_{i=1}^{n} \frac{(\partial/\partial\theta)q(x_i; \theta)}{q(x_i; \theta)} - \frac{n(\partial/\partial\theta)Q(x_n; \theta)}{Q(x_n; \theta)} = 0. \end{aligned} \tag{7}$$

Since the second equation cannot be solved analytically, there is no closed forms for $\hat{\beta}$ and $\hat{\theta}$. Some numerical methods can be employed such as the Newton–Raphson method.

2.2. Interval estimation

Denote $Y_i = \beta Q(X_i; \theta)$, it can be seen that $Y_i, i = 1, 2, \dots, n$, are record values from the standard exponential distribution with mean 1. Since the exponential distribution has the lack of memory property consequently the differences