Journal of Computational and Applied Mathematics 238 (2013) 14–28

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients

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ARTICLE INFO

Article history:
Received 26 January 2011
Received in revised form 14 August 2012

MSC:
65C30
65L20
60H10

Keywords:
Super-linear growth
Stochastic differential equation
Strong convergence
Backward Euler–Maruyama scheme
LaSalle principle
Almost sure stability

ABSTRACT

We are interested in the strong convergence and almost sure stability of Euler–Maruyama (EM) type approximations to the solutions of stochastic differential equations (SDEs) with non-linear and non-Lipschitzian coefficients. Motivation comes from finance and biology where many widely applied models do not satisfy the standard assumptions required for the strong convergence. In addition we examine the globally almost surely asymptotic stability in this non-linear setting for EM type schemes. In particular, we present a stochastic counterpart of the discrete LaSalle principle from which we deduce stability properties for numerical methods.

1. Introduction

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is to say, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \ldots, w_d(t))^T$ be a d-dimensional Brownian motion defined on the probability space, where T denotes the transpose of a vector or a matrix. In this paper we study the numerical approximation of the stochastic differential equations (SDEs)

$$dx(t) = f(x(t))dt + g(x(t))dw(t). \tag{1.1}$$

Here $x(t) \in \mathbb{R}^n$ for each $t \geq 0$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times d}$. For simplicity we assume that $x_0 \in \mathbb{R}^n$ is a deterministic vector. Although the method of Lyapunov functions allows us to show that there are solutions to a very wide family of SDEs (see e.g. [1,2]), in general, both the explicit solutions and the probability distributions of the solutions are not known. We therefore consider computable discrete approximations that, for example, could be used in Monte Carlo simulations. Convergence and stability of these methods are well understood for SDEs with Lipschitz continuous coefficients; see [3] for example. Our primary objective is to study classical strong convergence and stability questions for numerical approximations in the case where f and g are not globally Lipschitz continuous. A good motivation for our work is an instructive

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2012.08.015

conditional result of Higham et al. [4]. Under the local Lipschitz condition, they proved that uniform boundedness of moments of both the solution to (1.1) and its approximation are sufficient for strong convergence. That immediately raises the question of what type of conditions can guarantee such a uniform boundedness of moments. It is well known that the classical linear growth condition is sufficient to bound the moments for both the true solutions and their EM approximation [3,2]. It is also known that when we try to bound the moment of the true solutions, a useful way to relax the linear growth condition is to apply the Lyapunov-function technique, with $V(x) = \|x\|^2$. This leads us to the monotone condition [2]. More precisely, if there exist constants $\alpha, \beta > 0$ such that the coefficients of Eq. (1.1) satisfy

$$\langle x, f(x)\rangle + \frac{1}{2}\|g(x)\|^2 \leq \alpha + \beta\|x\|^2 \quad \text{for all } x \in \mathbb{R}^n, \tag{1.2}$$

then

$$\sup_{0\leq t\leq T} \mathbb{E}\|x(t)\|^2 < \infty \quad \forall T > 0. \tag{1.3}$$

Here, and throughout, $\|x\|$ denotes both the Euclidean vector norm and the Frobenius matrix norm and $\langle x, y\rangle$ denotes the scalar product of vectors $x, y \in \mathbb{R}^n$. However, to the best of our knowledge there is no result on the moment bound for the numerical solutions of SDEs under the monotone condition (1.2). Additionally, Hutzenthaler et al. [5] proved that in the case of super-linearly growing coefficients, the EM approximation may not converge in the strong L^p-sense nor in the weak sense to the exact solution. For example, let us consider a non-linear SDE

$$dx(t) = (\mu - \alpha x(t)^3)dt + \beta x(t)^2 dw(t), \tag{1.4}$$

where $\mu, \alpha, \beta \geq 0$ and $\alpha > \frac{1}{2}\beta^2$. In order to approximate SDE (1.4) numerically, for any Δt, we define the partition $\mathcal{P}_{\Delta t} := \{t_k = k\Delta t : k = 0, 1, 2, \dots, N\}$ of the time interval $[0, T]$, where $N\Delta t = T$ and $T > 0$. Then we define the EM approximation $Y_{t_k} \approx x(t_k)$ of (1.4) by

$$Y_{t_{k+1}} = Y_{t_k} + (\mu - \alpha Y_{t_k}^3)\Delta t + \beta Y_{t_k}^2 \Delta w_{t_k}, \tag{1.5}$$

where $\Delta w_{t_k} = w(t_{k+1}) - w(t_k)$. It was shown in [5] that

$$\lim_{\Delta t\to 0} \mathbb{E}\|Y_{t_N}\|^2 = \infty.$$

On the other hand, the coefficients of (1.4) satisfy the monotone condition (1.2) so (1.3) holds. Hence, Hutzenthaler et al. [5] concluded that

$$\lim_{\Delta t\to 0} \mathbb{E}\|x(T) - Y_{t_N}\|^2 = \infty.$$

It is now clear that to prove the strong convergence theorem under condition (1.2) it is necessary to modify the EM scheme. Motivated by the existing works [4,6] we consider implicit schemes. These authors have demonstrated that a backward Euler–Maruyama (BEM) method strongly converges to the solution of the SDE with one-sided Lipschitz drift and linearly growing diffusion coefficients. So far, to the best of our knowledge, most of the existing results on the strong convergence for numerical schemes only cover SDEs where the diffusion coefficients have at most linear growth [7–10,3]. However, the problem remains essentially unsolved for the important class of SDEs with super-linearly growing diffusion coefficients. We are interested in relaxing the conditions for the diffusion coefficients to justify Monte Carlo simulations for highly non-linear systems that arise in financial mathematics, [11–16], for example

$$dx(t) = (\mu - \alpha x^r(t))dt + \beta x^\rho(t)dw(t), \quad r, \rho > 1, \tag{1.6}$$

where $\mu, \alpha, \beta > 0$, or in stochastic population dynamics [17–21], for example

$$dx(t) = diag(x_1(t), x_2(t), \dots, x_n(t))[(b + Ax^2(t))dt + x(t)dw(t)], \tag{1.7}$$

where $b = (b_1, \dots, b_n)^T$, $x^2(t) = (x_1^2(t), \dots, x_n^2(t))^T$ and matrix $A = [a_{ij}]_{1\leq i,j\leq n}$ is such that $\lambda_{max}(A+A^T) < 0$, where $\lambda_{max}(A) = \sup_{x\in\mathbb{R}^n, \|x\|=1} x^T Ax$. The only results we know, where the strong convergence of the numerical approximations was considered for super-linear diffusion, is Szpruch et al. [22] and Mao and Szpruch [23]. In [22] authors have considered the BEM approximation for the following scalar SDE which arises in finance [13],

$$dx(t) = (\alpha_{-1}x(t)^{-1} - \alpha_0 + \alpha_1 x(t) - \alpha_2 x(t)^r)dt + \sigma x(t)^\rho dw(t) \quad r, \rho > 1.$$

In [23], this analysis was extended to the multidimensional case under specific conditions for the drift and diffusion coefficients. In the present paper, we aim to prove strong convergence under general monotone condition (1.2) in a multi-dimensional setting. We believe that this condition is optimal for boundedness of moments of the implicit schemes. The reasons that we are interested in the strong convergence are: (a) the efficient variance reduction techniques, for example, the multilevel Monte Carlo simulations [24], rely on the strong convergence properties; (b) both weak convergence [3] and pathwise convergence [25] follow automatically.