Journal of Computational and Applied Mathematics 237 (2013) 162–170

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On decomposition-based block preconditioned iterative methods for half-quadratic image restoration

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ARTICLE INFO

Article history:
Received 2 January 2012
Received in revised form 15 July 2012

Keywords:
Half-quadratic regularization
Image restoration
Newton method
Matrix decomposition
Preconditioner
PCG method

ABSTRACT

Image restoration is a fundamental problem in image processing. Except for many different filters applied to obtain a restored image in image restoration, a degraded image can often be recovered efficiently by minimizing a cost function which consists of a data-fidelity term and a regularization term. In specific, half-quadratic regularization can effectively preserve image edges in the recovered images and a fixed-point iteration method is usually employed to solve the minimization problem. In this paper, the Newton method is applied to solve the half-quadratic regularization image restoration problem. And at each step of the Newton method, a structured linear system of a symmetric positive definite coefficient matrix arises. We design two different decomposition-based block preconditioning matrices by considering the special structure of the coefficient matrix and apply the preconditioned conjugate gradient method to solve this linear system. Theoretical analysis shows the eigenvector properties and the spectral bounds for the preconditioned matrices. The method used to analyze the spectral distribution of the preconditioned matrix and the correspondingly obtained spectral bounds are different from those in the literature. The experimental results also demonstrate that the decomposition-based block preconditioned conjugate gradient method is efficient for solving the half-quadratic regularization image restoration in terms of the numerical performance and image recovering quality.

1. Introduction

Digital image restoration is an active research topic in various areas of applied sciences such as medical and astronomical imaging, film restoration, image and video coding. Usually, a common image degradation follows the model: a clean image $\mathbf{x} \in \mathbb{R}^{n^2}$ is observed in the presence of a spatially invariant blur matrix $\mathbf{A} \in \mathbb{R}^{n^2 \times n^2}$ and of an additive zero-mean Gaussian white noise $\boldsymbol{\eta} \in \mathbb{R}^{n^2}$ of standard deviation σ. Therefore, the observed image $\mathbf{b}$ is obtained as

$$\mathbf{b} = \mathbf{A}\mathbf{x} + \eta.$$

The aim of the image reconstruction is to obtain an estimate of the original image $\mathbf{x}$ from the observed image $\mathbf{b}$. This task can be mathematically settled by minimizing a cost function $\mathcal{J} : \mathbb{R}^{n^2} \to \mathbb{R}$. The function $\mathcal{J}$ usually consists of a data-fidelity term and a regularization term Φ that is weighted by a parameter $\beta > 0$. It can be precisely described as:

$$\hat{\mathbf{x}} = \min_{\mathbf{x} \in \mathbb{R}^{n^2}} J(\mathbf{x}),$$

$$\mathcal{J}(\mathbf{x}) = \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \beta \Phi(\mathbf{x}),$$

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doi:10.1016/j.cam.2012.07.025

where $\|\mathbf{Ax}-\mathbf{b}\|_2^2$ is the fidelity term, and the regularization term Φ is of the form

$$\Phi(\mathbf{x}) = \sum_{i=1}^{r} \phi(\mathbf{g}_i^T \mathbf{x}), \tag{1.1}$$

with $\phi : \mathbb{R} \to \mathbb{R}$ being a continuously differentiable function and $\mathbf{g}_i : \mathbb{R}^{n^2} \to \mathbb{R}$, $i = 1, \ldots, r$, being linear operators. Particularly, $\mathbf{g}_i$ is the first- or the second-order difference operator. Let $\mathbf{G}$ denote the $r \times n^2$ matrix whose ith row is $\mathbf{g}_i$. For the image $\mathbf{x}$, $\mathbf{Gx}$ describes the edges of the image $\mathbf{x}$ to some extent. Assume that

$$\mathbf{A} \not\equiv 0, \qquad \mathbf{G} \not\equiv 0, \qquad \phi \not\equiv 0 \quad \text{and} \quad \ker(\mathbf{A}^T\mathbf{A}) \cap \ker(\mathbf{G}^T\mathbf{G}) = \{0\}, \tag{1.2}$$

where $\ker(\cdot)$ denotes the kernel of the corresponding matrix. Clearly, this assumption guarantees that $\alpha_1\mathbf{A}^T\mathbf{A} + \alpha_2\mathbf{G}^T\mathbf{G}$ is a symmetric positive definite matrix provided both α_1 and α_2 are positive constants.

In this paper, the convex and edge-preserving potential function $\phi : \mathbb{R} \to \mathbb{R}$ is considered. Typical examples of such functions are:

$$\phi_1(t) = |t|^\alpha, \quad 1 < \alpha < 2, \tag{1.3}$$

$$\phi_2(t) = |t|/\alpha - \log(1 + |t|/\alpha), \tag{1.4}$$

$$\phi_3(t) = \sqrt{\alpha + t^2}, \tag{1.5}$$

$$\phi_4(t) = \log(\cosh(\alpha t))/\alpha, \tag{1.6}$$

$$\phi_5(t) = \begin{cases} t^2/(2\alpha), & \text{if } |t| \le \alpha, \\ |t| - \alpha/2, & \text{if } |t| > \alpha, \end{cases} \tag{1.7}$$

where $\alpha > 0$ in $\phi_j(t)$ $(j = 2, 3, 4, 5)$ is a prescribed parameter; see [1–4]. We consider the case that ϕ is convex, even, and is $\mathbb{C}^2$, and that

$$\mathbf{A}^T\mathbf{A} \text{ is invertible} \quad \text{and/or} \quad \phi''(t) > 0, \quad \forall t \in \mathbb{R}. \tag{1.8}$$

The computation of the minimizer $\hat{\mathbf{x}}$ of the cost-function $\mathcal{J}$ is very costly because of the nonlinearity of the function $\mathcal{J}$. In [5,3], an augmented cost function $\widetilde{\mathcal{J}} : \mathbb{R}^{n^2} \times \mathbb{R}^r \to \mathbb{R}$ that involves an auxiliary variable $\mathbf{v} \in \mathbb{R}^r$ in the following form is proposed

$$\widetilde{\mathcal{J}}(\mathbf{x}, \mathbf{v}) = \|\mathbf{Ax} - \mathbf{b}\|_2^2 + \beta \sum_{i=1}^{r} \left(\frac{1}{2}(\mathbf{g}_i^T\mathbf{x} - v_i)^2 + \psi(v_i) \right), \tag{1.9}$$

where

$$\phi(t) = \min_{s \in \mathbb{R}} \left\{ \frac{1}{2}(t - s)^2 + \psi(s) \right\}, \quad \forall t \in \mathbb{R}, \tag{1.10}$$

and $\psi : \mathbb{R} \to \mathbb{R}$ is a prescribed dual potential function that can be determined by using the theory of convex conjugacy, $\{v_i\}$ are the entries of the vector $\mathbf{v}$. We remark that ϕ is a potential function used as the regularization term in (1.1). The condition (1.10) ensures that

$$\mathcal{J}(\mathbf{x}) = \min_{\mathbf{v} \in \mathbb{R}^r} \widetilde{\mathcal{J}}(\mathbf{x}, \mathbf{v}), \quad \forall \mathbf{x} \in \mathbb{R}^{n^2}.$$

The regularization term involved in $\widetilde{\mathcal{J}}$ is half-quadratic, which is the reason of the method named. In [1], the minimizer $(\mathbf{x}, \mathbf{v})$ of $\widetilde{\mathcal{J}}$ is calculated by using alternating minimization. In order to speed up the convergence rate of the method, we may adopt the Newton-type method to solve (1.9). To this end, we revisit the Hessian of $\widetilde{\mathcal{J}}(\mathbf{x}, \mathbf{v})$, which is given by

$$\mathcal{H}(\mathbf{x}, \mathbf{v}) = \begin{bmatrix} 2\mathbf{A}^T\mathbf{A} + \beta\mathbf{G}^T\mathbf{G} & -\beta\mathbf{G}^T \\ -\beta\mathbf{G} & \beta\mathbf{I} + \beta\,\mathrm{diag}(\psi''(v_i)) \end{bmatrix} := \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{21} & \mathbf{H}_{22}(\mathbf{v}) \end{bmatrix}. \tag{1.11}$$

Here, $\mathbf{I}$ represents the identity matrix, and $\mathrm{diag}(\psi''(v_i))$ is a diagonal matrix whose diagonal entries are given by $\{\psi''(v_i)\}$.

Theorem 1.1 (*Theorem 1.1 in [4]*). *Under the assumptions in (1.8) and (1.2), the Hessian matrix $\mathcal{H}(\mathbf{x}, \mathbf{v})$ is symmetric positive definite for all $\mathbf{x}$ and $\mathbf{v}$.*

Therefore, in each Newton's iteration, the Newton equation leads to a structured symmetric positive definite linear system

$$\mathcal{H}(\mathbf{x}, \mathbf{v})\mathbf{d} = \mathbf{r}. \tag{1.12}$$

The main contribution of this paper is to speed up the iterative procedure by solving (1.12) using the *Preconditioned Conjugate Gradient* (PCG) method with decomposition-based preconditioners such that this kind of structured linear systems can be