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A composite Level Set and Extended-Domain-Eigenfunction Method for simulating 2D Stokes flow involving a free surface

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ABSTRACT

In this paper, the Extended-Domain-Eigenfunction-Method (EDEM) is combined with the Level Set Method in a composite numerical scheme for simulating a moving boundary problem. The liquid velocity is obtained by formulating the problem in terms of the EDEM methodology and solved using a least square approach. The propagation of the free surface is effected by a narrow band Level Set Method. The two methods both pass information to each other via a bridging process, which allows the position of the interface to be updated. The numerical scheme is applied to a series of problems involving a gas bubble submerged in a viscous liquid moving subject to both an externally generated flow and the influence of surface tension.

1. Introduction

In recent publications [1,2] the authors investigated a means of solving linear elliptic boundary value problems (EBVPs) on annular domains that are asymmetric. It is well known, of course, that problems involving domains with sufficient symmetry are amenable to the separation of variables method. In other cases, an EBVP would have to be tackled using numerical techniques such as the Finite Difference Method [3], the Finite Element Method [4,5] or the Boundary Element Method [5,6]. However, in [1,2] we explored the mathematical validity of a semi-analytic approach based on an eigenfunction expansion of the solution to a related problem on a larger domain having greater symmetry, the Extended-Domain Eigenfunction Method (EDEM).

Over the years there have been many like-minded, semi-analytic approaches, the Trefftz method [7,8] and its variants [9–11], being one class of methods. Independent of us, Shankar [12–15] espoused a method which lies closest to EDEM. However, to our knowledge none of the methods proposed previously were rigorously justified, nor have necessary and sufficient conditions for their application been given. In [1] we provided this justification and detailed necessary and sufficient conditions for the case of the scalar Laplace operator. In [2], we presented a numerical study, based on the scalar modified Helmholtz operator, comparing the EDEM with the Boundary Element Method [5,6] for both speed and accuracy. The comparison spoke clearly in favor of the EDEM. Such a numerical comparison of two very well suited techniques had not been undertaken before.

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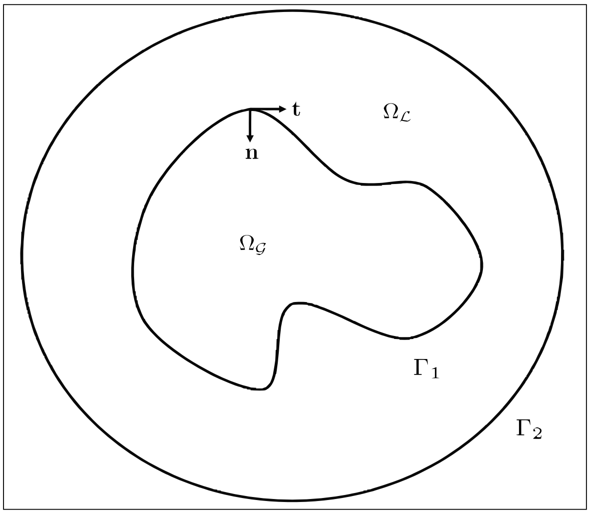

Fig. 1. 2D schematic of the physical domain Ω containing a liquid region $\Omega_{\mathcal{L}}$ and a gas region $\Omega_{\mathcal{G}}$ divided by the interface Γ_1.

In this paper we explore the EDEM further with an application to the Stokes vector elliptic partial differential equation system and, moreover, combine the EDEM with the Level Set Method (LSM). This composite method provides a fast and accurate algorithm with which to study two-phase hydrodynamic flow problems involving free boundaries. By application to two-dimensional systems we demonstrate that the method is able to capture quantitatively the expected behavior of a free surface moving under both surface tension and externally driven flow. Shankar [12–15] used a least square technique to study steady state hydrodynamic systems. We now consider a similar application but extend the proposition to include a study of the behavior of free interfaces under the action of hydrodynamic stresses.

The simulation of physical systems such as a bubble's free surface evolving under flow remains a problem of interest within the scientific community due to its complexity and diversity of behavior. In recent times, with the improvement of computer systems, there has also been a push for quick, efficient and physically realistic simulation techniques for industry based applications in computer graphics and animation [16,17]. The development of the LSM provided a means for both simulating and visualizing the evolution of free surface bodies. It now has an intimate connection with surface evolution. Earlier applications of the LSM, for example, included solving two phase flow problems [18]. Other efforts focused on using the LSM together with other techniques, which took responsibility for determining the velocity. For example, in recent publications the LSM was combined with the FEM [19] to simulate bubble clusters, while in [20] the BEM was incorporated to describe potential flow models of breaking waves over sloping beaches.

In the next section, we outline the problem we will use to demonstrate the application of the proposed LSM–EDEM composite scheme. In Section 3, an overview of the Level Set Method and mathematical principles of the EDEM are given. Numerical details on the implementation of the composite scheme are presented in Section 4. In the results section we present three two-dimensional examples of hydrodynamic problems of increasing complexity. The final section provides the summary conclusion that in each case the numerical findings are consistent with physical expectations and give encouragement to extending the composite method to three dimensional systems of practical interest.

2. The governing equations

We consider a two dimensional, bounded region Ω containing a liquid region $\Omega_{\mathcal{L}}$ and a region of gas, $\Omega_{\mathcal{G}}$ as depicted in Fig. 1. The liquid is in motion. Both the gas and liquid regions, $\Omega_{\mathcal{G}}$ and $\Omega_{\mathcal{L}}$, are time dependent while their union $\Omega = \Omega_{\mathcal{L}} \cup \Omega_{\mathcal{G}}$ remains constant. The moving interface between the liquid and gas region at time t will be denoted by the C^1 parametrization, $\Gamma_1(s, t) = (r(s; t), \theta(s; t))$, where t is time (in seconds) and s is the arc length (in meters). The outer boundary of Ω is fixed and denoted by Γ_2. It is assumed that the liquid is viscous, incompressible with no external body forces (*e.g.*, gravity) acting on it. We assume further that the fluid motion is slow enough to be described by the Stokes equations for creeping flow [21,22],

$$\begin{array}{l} \mu\nabla^2\mathbf{u} = \nabla P, \\ \nabla\cdot\mathbf{u} = 0, \end{array} \quad \mathbf{x} \in \Omega(t). \tag{1}$$

For convenience we employ polar coordinates since the study is confined to two dimensions. The liquid velocity vector has then components in the r and θ directions, $\mathbf{u}(r, \theta, t) = (\nu(r, \theta, t), \zeta(r, \theta, t))$, $P(r, \theta, t)$ is the pressure and μ is the dynamic viscosity.