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A class of boundary value problems for first-order impulsive integro-differential equations with deviating arguments☆

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ABSTRACT

This paper is concerned with a class of boundary value problems for nonlinear mixed impulsive integro-differential equations with deviating arguments. We establish a new comparison principle and use the method of upper and lower solutions together with the monotone iterative technique. Under suitable conditions, we obtain the existence results of extremal solutions for the problems. An example is also given to illustrate our results.

1. Introduction

Impulsive differential equations have become increasingly important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. There has been a significant development in impulse theory. In particular, there is an increasing interest in the study of nonlinear mixed integro-differential equations with deviating arguments and multipoint boundary value problems (BVPs) [1–4] for impulsive differential equations.

In [5], the method of lower and upper solutions combined with the monotone iterative technique and the numerical-analytic method were applied to study the problem

$$\begin{cases} x'(t) = f\left(t, x(t), \displaystyle\int_0^T k(s)x(s)ds\right) & t \in J = [0, T] \\ x(0) = \lambda x(T) + \displaystyle\int_0^T D(s)x(s)ds + d & d \in R, \end{cases}$$

where $f \in C[J \times R^2, R]$, f is non-decreasing with respect to the third variable, $k, D \in C[J, R_+]$, and $\lambda \geq 0$.

Chen and Shen [6] studied

$$\begin{cases} u'(t) = f(t, u(t), u(\theta(t))) & t \neq t_k,\ t \in J = [0, T] \\ \Delta u(t_k) = I_k(u(t_k)) & k = 1, 2, \ldots, m \\ u(0) + \mu \displaystyle\int_0^T u(s)ds = u(T), \end{cases}$$

where $f \in C[J \times R^2, R]$, and $\mu = 1$ or -1, by the method of upper and lower solutions and the monotone iterative technique.

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Motivated by the above, we are concerned with the following BVPs of nonlinear mixed impulsive integro-differential equations with deviating arguments:

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t)), Tu, Su) \quad t \neq t_k,\ t \in J = [0, T] \\ \Delta u(t_k) = I_k(u(t_k)) \quad k = 1, 2, \ldots, m \\ u(0) = \lambda_1 u(T) + \lambda_2 u(\eta) + \lambda_3 \displaystyle\int_0^T w(s, u(s))ds + k, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $I_k \in C(R, R)$, f is continuous everywhere except at $\{t_k\} \times R^4$; $f(t_k^+, \cdot, \cdot, \cdot, \cdot)$ and $f(t_k^-, \cdot, \cdot, \cdot, \cdot)$ exist with $f(t_k^-, \cdot, \cdot, \cdot, \cdot) = f(t_k, \cdot, \cdot, \cdot, \cdot)$;

$$(Tu)(t) = \int_0^{\beta(t)} k(t, s)u(\gamma(s))ds, \qquad (Su)(t) = \int_0^T h(t, s)u(\delta(s))ds,$$

and $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $w \in C(J \times R, R)$, $0 \le \lambda_1 \le 1$, $0 \le \lambda_2$, $0 \le \lambda_3$, $k \in R$, and $0 \le \eta \le T$. The assumptions concerning $\alpha, \beta, \gamma, \delta, k$, and h will be given latter. The boundary conditions in Eq. (1.1) involve several special cases such as periodic boundary conditions, anti-periodic boundary conditions, integral boundary, and initial problems.

Special cases

(i) If $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = k = 0$, then Eq. (1.1) reduces to the periodic boundary value problem (cf. [7–11]).
(ii) If $\lambda_2 = 1 + \lambda_1$, $\eta = 0$, and $\lambda_3 = k = 0$, then Eq. (1.1) reduces to the anti-periodic boundary value problem (cf. [12–15]).
(iii) If $\lambda_3 \neq 0$ and $\lambda_2 = k = 0$, then Eq. (1.1) reduces to the integral boundary value problems which have been studied in [16–19].
(iv) If $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then Eq. (1.1) reduces to initial problems (cf. [20,21]).

For example, if $\lambda_2 = 1 + \lambda_1$, $\eta = 0$, and $\lambda_3 = k = 0$, Eq. (1.1) reduces to an anti-periodic boundary value problem, which was considered by Wang and Zhang in [15]. There, the existence results of quasi-extremal solutions for the anti-periodic boundary value problems was obtained by the method of upper and lower solutions with the monotone iterative technique. Therefore, we extend some previous results in many respects.

The article is organized as follow. In Section 2, we establish a new comparison principle. In Section 3, by using of the monotone iterative technique and the method of upper and lower solutions, we obtain the existence results of extremal solutions for (1.1). In Section 4, we give an example that illustrates our results.

2. Preliminaries and lemmas

Let $PC(J) = \{x : J \to R; x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k) = x(t_k^-),\ k = 1, 2, \ldots, m\}$; $PC^1(J) = \{x \in PC(J) : x'(t)$ is continuous everywhere except for some t_k at which $x'(t_k^+)$ and $x'(t_k^-)$ exist and $x'(t_k) = x'(t_k^-),\ k = 1, 2, \ldots, m\}$. Let $J^- = J \setminus \{t_k,\ k = 1, 2, \ldots, m\}$; $PC(J)$ and $PC^1(J)$ are Banach spaces with the norms $\|x\|_{PC} = \sup\{|x(t)| : t \in J\}$ and $\|x\|_{PC^1} = \max\{\|x\|_{PC}, \|x'\|_{PC}\}$. $x \in PC^1(J)$ is called a solution of BVPs (1.1) if it satisfies Eq. (1.1).

In what follows, we need the following hypotheses.

(H_1) $\alpha, \beta, \gamma, \delta \in C(J, J)$, $N, K, H \in C(J, R_+)$, $k \in C(\Omega, R_+)$, $h \in C(J^2, R_+)$, $R_+ = [0, +\infty)$, $\Omega = \{(t, s) \in J^2 \mid 0 \le s \le \beta(t)\}$, $M \in C(J, R)$, $\int_0^T M(\tau)d\tau \ge 0$, and $0 \le L_k \le 1$, $0 < r \le 1$.
For convenience, we set

$$\begin{cases} N^*(t) = N(t)e^{\int_0^t M(s)ds}e^{-\int_0^{\alpha(t)} M(s)ds}, \qquad K^*(t) = K(t)e^{\int_0^t M(s)ds}, \\ H^*(t) = H(t)e^{\int_0^t M(s)ds}, \qquad k^*(t, s) = k(t, s)e^{-\int_0^{\gamma(s)} M(\tau)d\tau}, \\ h^*(t, s) = h(t, s)e^{-\int_0^{\delta(s)} M(\tau)d\tau}, \qquad r^* = re^{-\int_0^T M(s)ds}, \end{cases} \tag{2.1}$$

$\theta^*(t) = N^*(t) + K^*(t)\int_0^{\beta(t)} k^*(t, s)ds + H^*(t)\int_0^T h^*(t, s)ds \not\equiv 0$ for $t \in J$, $\mu^* = \int_0^T \theta^*(t)dt$.

(H_2)

$$\left[\mu^* + \sum_{k=1}^m L_k\right] \le r^*.$$

Lemma 2.1. *Assume that* (H_1) *and* (H_2) *hold, and that* $q \in PC^1(J)$ *such that*

$$\begin{cases} q'(t) \le -M(t)q(t) - (\mathscr{H}q)(t) \quad t \neq t_k,\ t \in J = [0, T] \\ \Delta q(t_k) \le -L_k(q(t_k)) \quad k = 1, 2, \ldots, m \\ q(0) \le rq(T), \end{cases} \tag{2.2}$$