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Exponentially-fitted methods and their stability functions

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ABSTRACT

We investigate the properties of stability functions of exponentially-fitted Runge–Kutta methods, and we show that it is possible (to some extent) to determine the stability function of a method without actually constructing the method itself. To focus attention, examples are given for the case of one-stage methods. We also make the connection with so-called integrating factor methods and exponential collocation methods. Various approaches are given to construct these methods.

1. Introduction

For the numerical integration of systems of first order ODEs

$$y' = f(x, y), \tag{1.1}$$

the family of Runge–Kutta methods is an excellent tool. In the case of initial value problems, the stability of such an RK method plays an important role and the stability properties of the methods should be examined. Therefore, the method is applied to the linear equation

$$y' = \lambda\, y \tag{1.2}$$

giving rise to a relation of the form $y_{n+1} = R(z)\, y_n$ with $z := \lambda\, h$ and whereby $R(z)$ is called the stability function of the method. For some families of RK methods the stability function can be written down without actually constructing the method: e.g.

- for an explicit $s \leq 4$ stage method of order s, $R(z) = \sum_{j=0}^{s} \frac{z^s}{s!}$,
- for an s-stage Gauss method, $R(z) = \hat{R}_s^s(z)$, where $\hat{R}_s^s(z)$ is the Padé-approximant of order $[s/s]$ of e^z,
- also for methods of Lobotto type and Radau type the function $R(z)$ is a Padé-approximant of e^z,
- ....

On the other hand, several authors have developed exponentially fitted Runge–Kutta (EFRK) methods. These methods are designed to solve problems whose solutions are weighted sums of polynomials and exponentials (or in the complex case trigonometric functions). So far, the properties of their stability functions have not been studied in detail. In this paper, the question is raised if it is possible, given the set of functions to which the method is fitted, to predict the exact expression of its stability function without actually constructing the method.

As we are mainly interested in the stability function of the methods, we will restrict ourselves in this paper to scalar problems. However, we want to emphasize that the methods that will be discussed, can also be reformulated to solve systems of equations.

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In the next section, we start with a short introduction to EFRK methods. In Section 3, we show that if the set of functions to be integrated exactly by the method is known, then also the conditions to impose on the stability function are known. In Section 4, we consider some examples of such stability functions of one stage methods. In Sections 5 and 6, we make a connection with some very special types of exponentially-fitted methods: so-called integrating factor methods and exponential-collocation methods. Both kinds of methods have been successfully applied in solving semi-linear systems. An excellent overview of exponential integrators of both families is given in [1]. Both types of methods start from a common idea, which will be very important in our discussion: the problem (1.1) is rewritten in the form

$$y' - \omega y = \tilde{f}(x, y) = f(x, y) - \omega y, \tag{1.3}$$

which can be rewritten as

$$(\mathrm{e}^{-\omega x} y)' = \mathrm{e}^{-\omega x} \tilde{f}(x, y). \tag{1.4}$$

2. Exponentially-fitted Runge–Kutta methods

Exponentially-fitted Runge–Kutta methods for the solution of a first order problem (1.1) have been discussed by several authors, e.g. [2–11]. The most general form of such a method is

$$y_{n+1} = \gamma\, y_n + h \sum_{i=1}^{s} b_i f(x_n + c_i h, Y_i)$$

whereby

$$Y_i = \gamma_i\, y_n + h \sum_{j=1}^{s} a_{ij} f(x_n + c_j h, Y_j), \quad i = 1, \dots, s.$$

With such a method, a generalized Butcher tableau can be associated:

$$\begin{array}{c|c|cccc} c_1 & \gamma_1 & a_{11} & \dots & a_{1s} \\ c_2 & \gamma_2 & a_{21} & \dots & a_{2s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_s & \gamma_s & a_{s1} & \dots & a_{ss} \\ \hline & \gamma & b_1 & \dots & b_s \end{array}$$

or

$$\begin{array}{c|c|c} c & \Gamma & A \\ \hline & \gamma & b^T \end{array}$$

The coefficients of these EFRK methods in general depend upon the product $z_0 := \omega h$ (some authors would explicitly denote A, b and c as $A(z_0)$, $b(z_0)$ and $c(z_0)$), where ω is a parameter that can be related to the solution of the problem to be solved. In fact, EF methods are designed to solve problems which have an exponential behaviour or (in the case ω is purely imaginary) a periodic behaviour. To construct such a EFRK method, a set of linear functionals can be introduced [12]:

$$\mathcal{L}_i[y(x); h] = y(x + c_i h) - \gamma_i\, y(x) - h \sum_{j=1}^{s} a_{ij} y'(x + c_j h), \quad i = 1, \dots, s$$

and

$$\mathcal{L}[y(x); h] = y(x + h) - \gamma\, y(x) - h \sum_{i=1}^{s} b_i y'(x + c_i h).$$

Next, conditions are imposed onto these functionals. For each stage of the method, a so-called fitting space is determined. In this paper, we will mainly consider the construction of implicit methods. In that case, each stage contains $s + 1$ parameters and for each stage the same fitting space $\mathcal{S}$ of dimension $s + 1$ can be considered.

It is well-known that collocation offers an alternative way to construct such methods: a function $P(x) \in \mathcal{S}$ is constructed such that

$$\begin{cases} P(x_n) = y_n \\ P(x_n + c_i h)' = f(x_n + c_i h, P(x_n + c_i h)), \quad i = 1, \dots, s. \end{cases} \tag{2.5}$$

The method is then defined by imposing $y_{n+1} := P(x_n + h)$.

Vanden Berghe et al. [7] have constructed methods for which

$$\mathcal{S} = \{x^q\, \mathrm{e}^{\pm\omega x} | q = 0, 1, \dots, P\} \cup \{x^q | q = 0, 1, \dots, K\}$$