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Numerical integration of functions with a very small significant support☆

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To the memory of our friend Donato Trigiante

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ABSTRACT

In some applications, one has to deal with the problem of integrating, over a bounded interval, a smooth function taking significant values, with respect to the machine precision or to the accuracy one wants to achieve, only in a very small part of the domain of integration. In this paper, we propose a simple and efficient numerical approach to compute or discretize integrals of this type. We also consider a class of second kind integral equations whose integral operator has the above behavior. Some numerical testing is presented.

1. Introduction

In some applications, one has to deal with the problem of integrating, over a bounded interval, a smooth function taking significant values, with respect to the machine precision or to the accuracy one wants to achieve, only in a very small part of the domain of integration. A couple of examples of this situation are given by the following integrals:

$$I_1 = \int_0^1 \frac{x}{\sqrt{1-x}} f(x)dx, \quad f(x) = \frac{e^{-(2x_1^2+x_2^2)}}{\sqrt{1+x}}, \qquad x_i = x_i(x), \tag{1}$$

$$I_2(x) = \int_\Gamma e^{-\frac{|x-y|^2}{4\epsilon}} f(y)d\Gamma_y, \quad x \in \Gamma \tag{2}$$

where I_1 is taken from [1], while in the second integral, which represents a heat potential-Gauss transform, Γ is a (open or closed) smooth curve in R^2 and $\epsilon > 0$ is a parameter whose values are (very) small.

In the applications where we have taken these two examples, it is crucial to approximate, with the needed accuracy, the integrals by means of simple quadrature rules using, if possible, a very low number of nodes. Since we are assuming that the functions $f(x)$ are very smooth, a natural approach is that of using proper Gaussian rules. However, in spite of the degree of smoothness of $f(x)$ in I_1, and of the whole integrand functions of I_2, the use of corresponding Gaussian rules would require an excessively high number of function evaluations. This is because the majorities of the quadrature nodes fall where the integrand functions take negligible values.

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In the next section we will present an alternative and more efficient approach to compute integrals of the type above. Then, in Section 3 we will use these quadrature rules to construct stable and convergent Nyström interpolants for the numerical solution of second kind integral equations, defined by integral operators whose integrands have a behavior like that described above.

2. A truncated Gaussian rule

For simplicity, we will consider the numerical evaluation of integrals of the form

$$\int_0^1 w(x)f(x)dx \tag{3}$$

where $w(x) = (1-x)^\alpha x^\beta,\ \alpha, \beta > -1$ is the Jacobi weight function, and $f(x)$ is smooth in $[0, 1]$ and decay exponentially to zero away from the endpoint $x = 0$. That is, $f(x)$ is considered significant, with respect to the machine precision or to the required accuracy, only in an interval of the type $[0, \tau)$, $0 < \tau < 1$, τ away from endpoint $x = 1$. In a more general situation the function $f(x)$ may have a two sided peak, or even more than one peak; however, by partitioning accordingly the interval of integration $(0, 1)$ we can always reduce the problem to the case considered above.

Therefore we assume that $|f(x)| \le \epsilon$, with $\epsilon > 0$ chosen by the user, in $[\tau, 1]$. The most natural approach to approximate the above integral is to use the Gauss–Jacobi rule

$$\int_0^1 w(x)f(x)dx = \sum_{i=1}^{n} \lambda_i f(x_i) + e_n(f) \tag{4}$$

where, here and in the following, its nodes are assumed to be ordered as follows: $0 < x_1 < \cdots < x_n < 1$. We recall that for the remainder $e_n(f)$ several estimates are well known (see, for example [2]). However, taking into account the assumption we have made on the behavior of $f(x)$, and the positivity of the coefficients λ_i (and their behavior with respect to the index i), it is equally natural to delete from the Gaussian quadrature sum all the terms referring to abscissas falling in the interval $[\tau, 1]$. That is, having defined

$$n_\tau = n_\tau(n) : x_{n_\tau} = \max\{x_i \le \tau\} \tag{5}$$

we replace (4) by

$$\int_0^1 w(x)f(x)dx = \sum_{i=1}^{n_\tau} \lambda_i f(x_i) + e_n^\tau(f) \tag{6}$$

where now we have

$$e_n^\tau(f) = \sum_{i=n_\tau+1}^{n} \lambda_i f(x_i) + e_n(f),$$

that is,

$$|e_n^\tau(f)| \le |e_n(f)| + \epsilon \sum_{i=n_\tau+1}^{n} \lambda_i < |e_n(f)| + \epsilon\mu_0$$

where we have set

$$\mu_0 = \int_0^1 w(x)dx = B(\alpha+1, \beta+1)$$

$B(x, y)$ being the well known Beta function (see [9]). Notice that in the Legendre case ($\alpha = \beta = 0$) we have $\mu_0 = 1$. Notice also that (see [9]) $n_\tau \to \infty$ as $n \to \infty$. Thus the truncated Gaussian rule has the same error behavior of the full Gaussian rule, except for an extra term of order ϵ, which can be chosen as small as we like, by taking τ sufficiently close to $x = 1$.

Of course, if the order of magnitude of the maximum of $|f(x)|$ in $[0, \tau]$ is very different from 1, one should determine first an estimate (M_f) of this value and then replace above ϵ by ϵM_f.

The truncated Gaussian rule strategy is not a new idea. We have applied it in the case of Gauss–Laguerre formulas, to approximate integrals defined on $(0, \infty)$ whose integrand functions decay exponentially at infinity (see [3,4]). It has been applied also to the Gauss–Hermite case (see [5]).

A simpler and obvious approach to approximate integral (3), alternative to the truncation one, is to write first

$$\int_0^1 w(x)f(x)dx \approx \int_0^\tau w(x)f(x)dx \tag{7}$$