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Node-pair finite volume/finite element schemes for the Euler equation in cylindrical and spherical coordinates

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ABSTRACT

A numerical scheme is presented for the solution of the compressible Euler equations in both cylindrical and spherical coordinates. The unstructured grid solver is based on a mixed finite volume/finite element approach. Equivalence conditions linking the node-centered finite volume and the linear Lagrangian finite element scheme over unstructured grids are reported and used to devise a common framework for solving the discrete Euler equations in both the cylindrical and the spherical reference systems. Numerical simulations are presented for the explosion and implosion problems with spherical symmetry, which are solved in both the axial–radial cylindrical coordinates and the radial–azimuthal spherical coordinates. Numerical results are found to be in good agreement with one-dimensional simulations over a fine mesh.

1. Introduction

Diverse compressible flow fields of both industrial and academic interest exhibit relevant cylindrical or spherical symmetries, as is the case for example for astrophysical flows, nozzle flows, inertial confinement fusion (ICF) applications, sonoluminescence studies and nuclear explosions [1].

The numerical solution of the flow equations in these cases can be greatly simplified by expressing both the governing equations and the initial/boundary data within either a cylindrical or a spherical coordinate system. Indeed, it is often possible to reduce the three-dimensional problem to a simpler two-dimensional or even one-dimensional one, as is the case for example for cylindrically or spherically symmetric explosions. In the fifties, Goldstine and von Neumann [2] and Brode [3] used an artificial viscosity approach to solve the one-dimensional Euler equations in spherical coordinates for the explosion problem. Payne [4] used instead a finite difference approximation. In the seventies, Sod [5] applied for the first time a solution technique based on Riemann solvers. In 1999, Liu and collaborators [6] used the total variation diminishing (TVD) technique of Harten [7], in which the numerical scheme is obtained as a suitable combination of a high order scheme with a low order one to be used close to flow discontinuities.

For multidimensional problems, such as for example axisymmetric problems in which the solution is independent from the azimuthal coordinate in a cylindrical reference system, additional difficulties arise due to the singularity of the cylindrical coordinates. In this case, it is standard practice to either approximate the singular terms with *ad hoc* procedures [8–10] or resort to complex geometrical calculations of the control volume centroids [11].

In the present paper, a mixed finite volume (FV)/finite element (FE) approach [12] is used to derive a numerical scheme capable of solving the Euler equations for a compressible fluid in both cylindrical and spherical coordinate systems.

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The combined use of the finite volume and finite element techniques is made possible by the introduction of equivalence conditions that relate the FV metrics, i.e. cell volumes and integrated normals, to suitably defined FE integrals. Equivalence conditions relating FV and FE schemes have been derived for Cartesian coordinates in two and three spatial dimensions [13,14] and for cylindrical coordinates in axially symmetric two-dimensional problems [15]. In both cited references, equivalence conditions are obtained by neglecting higher order FE contributions. Subsequently in [16], equivalence conditions for the cylindrical coordinates have been derived for the first time without introducing any approximation into the FE discrete expression for the divergence operator. In [17,18], the spherical coordinate system is also considered and equivalence conditions are derived in this case. In [19] the differences between the consistent scheme and an alternative one violating the equivalence conditions have been quantified for the case of one-dimensional problems in cylindrical and spherical coordinates. An advantage of the proposed hybrid approach lies in the automatic treatment of the singularities of each reference system, namely, the axis and the origin in the cylindrical and spherical reference systems, respectively.

In the present paper the consistent formulation is presented in a unifying framework for both the cylindrical and spherical reference systems. Numerical results for the implosion and the explosion problem are also provided to demonstrate the correctness of the proposed approach and to compare the two formulations. In Sections 2 and 3 the spatial discretizations of the scalar equation using FE and FV approaches are presented, in cylindrical and spherical coordinates, respectively, for a model scalar conservation law. Conditions for equivalence between FE and FV metrics are also shown. In Section 5, numerical simulations are presented for the implosion and the explosion problem in cylindrical coordinates on the R–θ and Z–R (axisymmetric) planes and in the spherical coordinates on the r–ϕ plane. Numerical results are also compared to one-dimensional simulations. In Section 6 final remarks and comments are given.

2. The node-pair finite volume/element scheme in cylindrical coordinates

The node-pair finite volume/element scheme in cylindrical coordinates is now derived for a scalar conservation law. In the three-dimensional cylindrical reference system the model equation reads

$$\frac{\partial u}{\partial t} + \frac{\partial f_Z}{\partial Z} + \frac{1}{R}\frac{\partial}{\partial R}(Rf_R) + \frac{1}{R}\frac{\partial f_\theta}{\partial \theta} = 0, \tag{1}$$

where t is the time, Z, R and θ are the axial, radial and azimuthal coordinates, respectively, $u = u(Z, R, \theta,\ t)$ is the scalar unknown and $\boldsymbol{f}^{\varnothing}(u) = (f_Z, f_R, f_\theta)$ is the so-called flux function. A more compact form of the above equation is obtained by introducing the divergence operator in three-dimensional cylindrical coordinates $\boldsymbol{\nabla}^{\varnothing}\boldsymbol{\cdot}(\cdot)$ as follows:

$$\frac{\partial (u)}{\partial t} + \boldsymbol{\nabla}^{\varnothing}\boldsymbol{\cdot}\boldsymbol{f}^{\varnothing}(u) = 0. \tag{2}$$

2.1. Node-pair finite element discretization

The scalar conservation law (2) is now written in a weak form by multiplying it by the radial coordinate R and by a suitable Lagrangian test function $\phi_i \in V_h \subset H^1(\Omega)$. Integrating over the support Ω_i of φ_i gives

$$\int_{\Omega_i} R\varphi_i \frac{\partial u}{\partial t}\,\mathrm{d}\Omega^{\varnothing} + \int_{\Omega_i} R\varphi_i \boldsymbol{\nabla}^{\varnothing}\boldsymbol{\cdot}\boldsymbol{f}^{\varnothing}(u)\,\mathrm{d}\Omega^{\varnothing} = 0, \quad \forall i \in \mathcal{N}, \tag{3}$$

where $\mathcal{N}$ is the set of all nodes of the triangulation. Note that on multiplying by R, the numerical singularity of the cylindrical reference system is formally removed [15]. In the following, to simplify the notation, the infinitesimal volume $\mathrm{d}\Omega = R\,\mathrm{d}R\,\mathrm{d}\theta\,\mathrm{d}Z$ is not indicated in the integrals. Integrating by parts immediately gives

$$\int_{\Omega_i} R\varphi_i \frac{\partial u}{\partial t}\,\mathrm{d}\Omega^{\varnothing} = \int_{\Omega_i} R\boldsymbol{f}^{\varnothing}\boldsymbol{\cdot}\boldsymbol{\nabla}^{\varnothing}\varphi_i\,\mathrm{d}\Omega^{\varnothing} + \int_{\Omega_i} \varphi_i\boldsymbol{f}^{\varnothing}\boldsymbol{\cdot}\boldsymbol{\nabla}^{\varnothing}R\,\mathrm{d}\Omega^{\varnothing} - \int_{\partial\Omega_i^{\partial}} R\varphi_i\boldsymbol{f}^{\varnothing}\boldsymbol{\cdot}\boldsymbol{n}_i^{\varnothing}\,\mathrm{d}\partial\Omega^{\varnothing} \tag{4}$$

where $\partial\Omega_i^{\partial} = \partial\Omega_i \cap \partial\Omega$, with $\partial\Omega_i$ and $\partial\Omega$ the boundary of Ω_i and that of the computational domain Ω, respectively, and where $\boldsymbol{n}_i^{\varnothing} = n_Z\hat{\boldsymbol{Z}} + n_R\hat{\boldsymbol{R}} + n_\theta\hat{\boldsymbol{\theta}}$ is the outward versor normal to Ω_i. The scalar unknown is now interpolated as $u(Z, R, \theta, t) \simeq u_h(Z, R, \theta, t) = \sum_{k\in\mathcal{N}} u_k(t)\varphi_k(Z, R, \theta)$, to obtain the Bubnov–Galerkin approximation of (1), namely

$$\sum_{k\in\mathcal{N}_i} \frac{\mathrm{d}u_k}{\mathrm{d}t} M_{ik}^{\varnothing} = \int_{\Omega_i} R\boldsymbol{f}^{\varnothing}(u_h)\boldsymbol{\cdot}\boldsymbol{\nabla}^{\varnothing}\varphi_i\,\mathrm{d}\Omega^{\varnothing} + \int_{\Omega_i} \varphi_i\boldsymbol{f}^{\varnothing}(u_h)\boldsymbol{\cdot}\boldsymbol{\nabla}^{\varnothing}R\,\mathrm{d}\Omega^{\varnothing} - \int_{\partial\Omega_i^{\partial}} R\varphi_i\boldsymbol{f}^{\varnothing}(u_h)\boldsymbol{\cdot}\boldsymbol{n}^{\varnothing}\,\mathrm{d}\partial\Omega^{\varnothing}, \tag{5}$$

where $\mathcal{N}_i$ is the set of shape functions φ_k whose support Ω_k overlaps Ω_i of φ_i and where

$$M_{ik}^{\varnothing} \stackrel{\text{def}}{=} \int_{\Omega_{ik}} R\varphi_i\varphi_k\,\mathrm{d}\Omega^{\varnothing},$$